

Solitons, Tau-functions and Hamiltonian Reduction for Non-Abelian Conformal Affine Toda Theories

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ABSTRACT

We consider the Hamiltonian reduction of the two-loop Wess-Zumino-Novikov-Witten model (WZNW) based on an untwisted affine Kac-Moody algebra $\hat{\mathcal{G}}$. The resulting reduced models, called *Generalized Non-Abelian Conformal Affine Toda (G-CAT)*, are conformally invariant and a wide class of them possesses soliton solutions; these models constitute non-abelian generalizations of the Conformal Affine Toda models. Their general solution is constructed by the Leznov-Saveliev method. Moreover, the dressing transformations leading to the solutions in the orbit of the vacuum are considered in detail, as well as the τ -functions, which are defined for any integrable highest weight representation of $\hat{\mathcal{G}}$, irrespectively of its particular realization. When the conformal symmetry is spontaneously broken, the G-CAT model becomes a generalized Affine Toda model, whose soliton solutions are constructed. Their masses are obtained exploring the spontaneous breakdown of the conformal symmetry, and their relation to the fundamental particle masses is discussed.

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1 Introduction

The study of classical and quantum non-linear integrable models in 1+1 dimensions is of great interest in High Energy Physics, where such models have been used as laboratories to develop methods to explore the non-linear perturbative aspects of gauge theories, gravity and string theory. In particular, they could help in understanding some stable classical solutions, like monopoles, which must have an important role in the quantum theory, and which cannot be understood by the existing methods.

In this paper, we construct the Generalized Non-Abelian Conformal Affine Toda models, the G-CAT models, which are a class of generalizations of the well known non-abelian Affine Toda Field theories that, being integrable and conformally invariant, provide one of the simplest example of mass generation through spontaneous symmetry breaking. To be precise, within these models, the mass of the solitons of the Affine Toda Theories can be understood in terms of a Higgs-like mechanism associated to the spontaneous breakdown of the conformal symmetry. Even more, within this approach, it is possible to put in one-to-one correspondence the solitons of the non-abelian Affine Toda models with its *massive* fundamental particles; in particular, their masses are proportional.

Within the integrable models in 1+1 dimensions, the investigation of the different Toda Field Theories has recently received a lot of attention. According to their underlying algebraic structure, they can be divided into three categories; each one exhibiting nice characteristic properties. First, associated to the finite simple Lie algebras, there are the Conformal Toda models, which are conformally invariant 1+1 field theories. Even more, they permit the construction of extensions of the Virasoro algebra including higher spin generators, namely W-algebras. The second class of theories are the Affine Toda models, based on loop algebras, which can be regarded as a perturbed Conformal Toda model where the conformal symmetry is broken by the perturbation while the integrability is preserved [1]. One of their main properties is that they possess soliton solutions. These two classes of models are called abelian or non-abelian referring to whether their fields live on an abelian or non-abelian group [2, 3, 4, 5]. Finally, the conformal symmetry can be restored in the abelian Affine Toda models just by adding two extra fields which do not modify the dynamics of the original model; one of these fields is a connection whose only role is to implement the conformal invariance. These are the so called Conformal Affine Toda models [6, 7], and they are based on a full Kac-Moody algebra; moreover, they are integrable [8], and have soliton solutions [9]. In fact, many properties of the Affine Toda models can be more easily understood by considering them as the Conformal Affine Toda models with the conformal symmetry spontaneously broken. All these Toda models can be obtained via Hamiltonian reductions of WZNW models [10, 6], whilst their one dimensional versions can be obtained from free motion on symmetric spaces [11, 12].

The G-CAT models consist on a WZNW model where the field takes values in a finite—not necessarily semisimple—Lie group G_0 , and where the symmetries of left and right translations by elements of G_0 are broken by a term of the form

$$\int d^2x \operatorname{Tr} \left(\Lambda_{-l} B \Lambda_l B^{-1} \right), \quad B \in G_0 \quad (1.1)$$

where $\Lambda_{\pm l}$ are constant elements of subspaces of grade $\pm l$ of a Kac-Moody algebra $\hat{\mathcal{G}}$; the

grades corresponding to a gradation of $\hat{\mathcal{G}}$ where the subspace of zero grade is the Lie algebra of G_0 . Notice that we allow the central term of $\hat{\mathcal{G}}$ and the grading operator to be generators of G_0 ; this leads to the conformal invariance of the G-CAT models, which generalizes the CAT models [6, 7] for non-abelian groups.

In section 2, we obtain the G-CAT models via Hamiltonian reduction of the two-loop WZNW model, where the fields are elements of a Kac-Moody group and the corresponding current algebra is a two-loop Kac-Moody algebra [6, 13]. The Hamiltonian reduction can be characterized by the choice of a gradation of the Kac-Moody algebra $\hat{\mathcal{G}}$ and of the elements $\Lambda_{\pm l}$ appearing in (1.1); the reduction can be performed for a general gradation, but we restrict ourselves to the case of integer gradations. The constraints are imposed on some non-chiral quantities which have a simpler algebraic structure than the currents. Actually, the generators $\Lambda_{\pm l}$ correspond to the constant value of those quantities in the subspaces of grade $\pm l$. When l is greater than the lowest positive grade, the constraints on the currents are of second class. However, not all the components of the currents with grade between zero and l are constrained, implying that the field B is not the only degree of freedom of the reduced model. The extra degrees of freedom correspond to two sets of chiral fields, one set for each chirality, but these chiral fields are decoupled from the field B .

In section 3, we show that the reduced model is conformally invariant because the current associated to the grading operator can be used to improve the stress tensor allowing the constraints to weakly commute with it. We also show that the field in the direction of the grading operator, called η , is free; even more, its interaction with the other fields is such that for every regular solution of η one can eliminate it from the remaining equations of motion by a coordinate transformation. Then, the resulting model is not conformally invariant and we refer to it as the Generalized non-abelian Affine Toda (G-AT). All this is a generalization of what occurs in the abelian Conformal Affine Toda models [14].

The general solution of the G-CAT models is constructed in section 4 using the method of Leznov and Saveliev [2, 3, 15]. For l not being the lowest positive grade, the number of chiral parameters of the solution is not equal to twice the dimension of G_0 . There is a number of extra parameters equal to the number of chiral fields appearing in the Hamiltonian reduction described above. Those parameters are present even though we start from a zero curvature connection depending only on the field B ; their origin has to do just with the algebraic structure of the connections.

The symmetries of the G-CAT models are discussed in section 5; they correspond to the left (right) translations on the Kac-Moody group which leave Λ_{-l} (Λ_l) invariant. One of our main interests is to construct the one-soliton solutions of the model, which are static solutions in some particular Lorentz frame. We show that the condition for the existence of static solutions is that $\Lambda_{\pm l}$ can be transformed by some constant element of G_0 into some elements $E_{\pm l}$ such that $[E_l, E_{-l}]$ is in the direction of the central term only. This condition is easily satisfied if $E_{\pm l}$ lie on a Heisenberg subalgebra of $\hat{\mathcal{G}}$.

The fundamental particles of the G-CAT model are obviously massless due to conformal invariance. In section 6 we show that the classical masses of the fundamental particles of the G-AT models appear as a consequence of the spontaneous breakdown of the conformal symmetry, resembling very much the Higgs mechanism in gauge theories. We show that the masses are proportional to the eigenvalues of the operator $[E_l, [E_{-l}, \cdot]]$ when acting on the

Lie algebra of G_0 ; the proportionality constant being the vacuum expectation value of the (Higgs-like) field $e^{l\eta}$.

In section 7 we use the dressing transformation method [16, 17, 18, 19] to construct the solutions of the G-CAT model lying in the orbit of the vacuum. The construction is quite simple and leads to a very useful expression for the solutions. In fact, it allows us to show that the *solitonic specialization* of the Leznov-Saveliev solution described in [20, 21] leads to all solutions in the orbit of the vacuum.

The results obtained with the dressing method guide us to the introduction of the τ -function [22, 23] in section 8. We define the τ -functions for the G-CAT models as the orbit of the highest weight state of an integrable representation of the Kac-Moody $\hat{\mathcal{G}}$ obtained under the action of the group element used in the dressing transformation. The particular integrable representation is the one associated to the gradation of $\hat{\mathcal{G}}$ used in the Hamiltonian reduction, in a manner explained in the appendix B. One of the nice features of our definition is that it is independent of the way the integrable representation is realized; in particular, it is independent of the level of the representation. In this sense, our results allow the generalization of the connection between τ -functions and zero-curvature integrable hierarchies of partial differential equations worked out in [24].

The masses of the one-soliton solutions are calculated in section 9. Following [9], we explore the conformal invariance of the G-CAT model to show that the masses of the solitons of the G-AT models come from a total divergence which is given by the improvement term of the G-CAT stress tensor. We find an explicit formula for those masses by using the highest weight state of an integrable representation of $\hat{\mathcal{G}}$ that is dual to the one used in the τ -function definition. Then, the one-soliton solutions can be put in one to one correspondence with the massive fundamental particles. In addition, non-vanishing soliton masses are proportional to the masses of their corresponding fundamental particles. This is a generalization of what occurs in the abelian Affine Toda models [9, 20], and suggests some deep structure in such theories which is still not well understood; they could be related to a *duality* transformation similar to the one conjectured in four-dimensional non-abelian gauge theories [25, 26].

In section 10 we discuss the G-CAT models associated to the principal and homogeneous gradations of any Kac-Moody algebra $\hat{\mathcal{G}}$. In the homogeneous case, the one-soliton solutions are studied in great detail, also considering the particular case when G_0 is compact. Finally, we present our conclusions in section 11.

In appendix A we show how to use the structure of the two-loop Kac-Moody current algebra to construct what we call the two-loop Virasoro algebra. In order to do that, we have to impose periodic boundary conditions on the currents, and the structure of such algebra is more complex when the two central terms of the two-loop current algebra are incommensurable; the role of this algebra in relation to the symmetries of the G-CAT model has not been investigated yet. Finally, appendices B and C include our conventions and some particular properties of Kac-Moody algebras used along the paper.

2 Generalized Reduction of Two-Loop WZNW model

The two-loop WZNW model was introduced in [6] as the generalization of the ordinary WZNW model to the affine case. Its equations of motion are given by

$$\partial_+ (\partial_- \hat{g} \hat{g}^{-1}) = 0 \quad ; \quad \partial_- (\hat{g}^{-1} \partial_+ \hat{g}) = 0 \quad (2.1)$$

where ∂_{\pm} are derivatives with respect to the light-cone variables $x_{\pm} = x \pm t$, and \hat{g} is an element of the group G formed by exponentiating an untwisted affine (real) Kac-Moody (KM) algebra $\hat{\mathcal{G}}$. Its generators T_a^m , D , and C satisfy the commutation relations

$$[T_a^m, T_b^n] = f_{ab}^c T_c^{m+n} + m C g_{ab} \delta_{m+n,0} \quad (2.2)$$

$$[D, T_a^m] = m T_a^m, \quad [C, D] = [C, T_a^m] = 0 \quad (2.3)$$

where f_{ab}^c are the structure constants of a finite (real) semisimple Lie algebra \mathcal{G} , n and m are integers, and g_{ab} is the Killing form of \mathcal{G} , *i.e.*, $g_{ab} = \text{Tr}(T_a T_b)$, T_a being the generators of \mathcal{G} . The non-degenerate bilinear form of $\hat{\mathcal{G}}$ is defined as (see also (B.7)-(B.9))

$$\begin{aligned} \text{Tr}(T_a^m T_b^n) &= \delta_{m+n,0} \text{Tr}(T_a T_b), & \text{Tr}(C D) &= 1 \\ \text{Tr}(C T_a^m) &= \text{Tr}(D T_a^m) = 0, \end{aligned} \quad (2.4)$$

and we will use the same notation, Tr , for both the Killing form of \mathcal{G} and the bilinear form of $\hat{\mathcal{G}}$.

The two-loop WZNW model is invariant under left and right translations

$$\hat{g}(x_+, x_-) \rightarrow \hat{g}_L(x_-) \hat{g}(x_+, x_-) \quad , \quad \hat{g}(x_+, x_-) \rightarrow \hat{g}(x_+, x_-) \hat{g}_R(x_+) \quad (2.5)$$

The corresponding Noether currents are the components of $\partial_- \hat{g} \hat{g}^{-1}$ and $\hat{g}^{-1} \partial_+ \hat{g}$, and they generate two commuting copies of the so called two-loop Kac-Moody algebra [6], defined by the relations

$$[J_a^m(x), J_b^n(y)] = f_{ab}^c J_c^{m+n}(x) \delta(x-y) + g_{ab} \delta_{m,-n} (k \partial_x \delta(x-y) + m J^C(x) \delta(x-y)) \quad (2.6)$$

$$[J^D(x), J_a^m(y)] = m J_a^m(y) \delta(x-y) \quad (2.7)$$

$$[J^C(x), J^D(y)] = k \partial_x \delta(x-y) \quad (2.8)$$

$$[J^C(x), J_a^m(y)] = 0 \quad (2.9)$$

The left and right currents satisfying the above relations are related to the group element \hat{g} in eq.(2.1) by

$$\mathcal{J}_R(x_+) = k \hat{g}^{-1} \partial_+ \hat{g} = \sum_{a,b} \sum_{n=-\infty}^{\infty} g^{ab} J_{R,a}^{-n}(x_+) T_b^n + J_R^D(x_+) C + J_R^C(x_+) D \quad (2.10)$$

$$\mathcal{J}_L(x_-) = -k \partial_- \hat{g} \hat{g}^{-1} = \sum_{a,b} \sum_{n=-\infty}^{\infty} g^{ab} J_{L,a}^{-n}(x_-) T_b^n + J_L^D(x_-) C + J_L^C(x_-) D \quad (2.11)$$

where g^{ab} is the inverse of the Killing form g_{ab} defined above. The different meaning of the two central extensions in eqs.(2.6)-(2.9) algebra is clarified by expressing the algebra as

$$[\text{Tr}(U \mathcal{J}(x)), \text{Tr}(V \mathcal{J}(y))] = \text{Tr}([U, V] \mathcal{J}(x)) \delta(x-y) + k \text{Tr}(UV) \partial_x \delta(x-y), \quad (2.12)$$

where U, V are two elements of the Kac-Moody algebra $\hat{\mathcal{G}}$, \mathcal{J} is either \mathcal{J}_R or \mathcal{J}_L , and Tr is the invariant bilinear form of $\hat{\mathcal{G}}$.

Consider now a gradation of the Kac-Moody algebra $\hat{\mathcal{G}}$

$$\hat{\mathcal{G}} = \bigoplus_s \hat{\mathcal{G}}_s \quad (2.13)$$

with

$$[\hat{\mathcal{G}}_s, \hat{\mathcal{G}}_r] \subset \hat{\mathcal{G}}_{s+r} \quad (2.14)$$

The reduction presented in this section does not require that this gradation is integer; it just needs that the grades s take zero, positive and negative values, *i.e.*,

$$\hat{\mathcal{G}} = \hat{\mathcal{G}}_+ \oplus \hat{\mathcal{G}}_0 \oplus \hat{\mathcal{G}}_- \quad (2.15)$$

with

$$\hat{\mathcal{G}}_+ \equiv \bigoplus_{s>0} \hat{\mathcal{G}}_s, \quad \hat{\mathcal{G}}_- \equiv \bigoplus_{s<0} \hat{\mathcal{G}}_s \quad (2.16)$$

Nevertheless, in the rest of the paper, we shall restrict ourselves to integer gradations, which can be described in a very systematic way using the results of [27, 28] (see appendix B).

We now consider those group elements that can be written in a ‘‘Gauss decomposition’’ form

$$\hat{g} = NBM \in G \quad (2.17)$$

where N, B and M are group elements formed by exponentiating elements of $\hat{\mathcal{G}}_+, \hat{\mathcal{G}}_0$ and $\hat{\mathcal{G}}_-$ respectively. It is well known that the use of the Gauss decomposition requires that the group has definite properties regarding its compactness. Nevertheless, when the subspaces of $\hat{\mathcal{G}}$ with opposite grades are non-degenerately paired by the bilinear form, as it happens when the gradation is integer, the validity of the Gauss decomposition does not imply any condition regarding the compactness of the subgroup formed by exponentiating the elements of $\hat{\mathcal{G}}_0, G_0$; it only requires that the restriction of the bilinear form to $\hat{\mathcal{G}}_+ \oplus \hat{\mathcal{G}}_-$ is maximally non-compact. When the gradation is integer (see (B.14)-(B.15)), this means that $\{H_1^n, \dots, H_r^n\}$, for all $n \neq 0$, and $\{E_\alpha^m\}$, with $m N_s + \alpha \cdot H_s \neq 0$, must be among the generators of the chosen real form of the algebra $\hat{\mathcal{G}}$. Then, one is free to choose the subgroup G_0 to be either compact or non-compact; each choice leading to different theories. In section 5, we shall come back to this point, which is very important in relation to the physical interpretation of the resulting models [29].

Using eq.(2.17), we can write the equations of motion (2.1) as

$$\partial_- K_R = -[K_R, \partial_- M M^{-1}] \quad (2.18)$$

$$\partial_+ K_L = [K_L, N^{-1} \partial_+ N] \quad (2.19)$$

where we have introduced

$$\begin{aligned} K_L &\equiv N^{-1} \partial_- \hat{g} \hat{g}^{-1} N \\ &= N^{-1} \partial_- N + \partial_- B B^{-1} + B \partial_- M M^{-1} B^{-1} \end{aligned} \quad (2.20)$$

$$\begin{aligned} K_R &\equiv M \hat{g}^{-1} \partial_+ \hat{g} M^{-1} \\ &= B^{-1} N^{-1} \partial_+ N B + B^{-1} \partial_+ B + \partial_+ M M^{-1} \end{aligned} \quad (2.21)$$

Although the quantities $K_{L/R}$ are not chiral, they have a simpler structure than the currents and will be very useful in what follows. We will reduce the two-loop WZNW model by imposing constraints not directly on the currents but on $K_{L/R}$. We impose the constraints

$$B^{-1} (N^{-1} \partial_+ N) B = \Lambda_l \quad (2.22)$$

$$B (\partial_- M) M^{-1} B^{-1} = \Lambda_{-l} \quad (2.23)$$

where $\Lambda_{\pm l}$ are constant elements of $\hat{\mathcal{G}}_{\pm l}$ ($l > 0$). These constraints reduce the two-loop WZNW model to a theory containing only the fields corresponding to the components of B and to the components of N and M associated to the generators whose grade is $< l$ and $> -l$ respectively.

To obtain the equations of motion for such model one notices that the constraints (2.22) and (2.23) imply that

$$N^{-1} \partial_+ N \in \hat{\mathcal{G}}_l \quad (2.24)$$

$$(\partial_- M) M^{-1} \in \hat{\mathcal{G}}_{-l} \quad (2.25)$$

Therefore the only terms of zero grade on the right hand side of (2.19) are coming from $[\Lambda_{-l}, N^{-1} \partial_+ N] = [\Lambda_{-l}, B \Lambda_l B^{-1}]$. So, we get

$$\partial_+ (\partial_- B B^{-1}) = [\Lambda_{-l}, B \Lambda_l B^{-1}] \quad (2.26)$$

which can also be written as

$$\partial_- (B^{-1} \partial_+ B) = -[\Lambda_l, B^{-1} \Lambda_{-l} B] \quad (2.27)$$

These are the equations of motion of what we call the *generalized non-abelian conformal affine Toda models* (G-CAT). When the gradation is integral and $l = 1$, these equations were first considered in [2, 3], but not including the field in the direction of the derivation D , which leads to the conformal invariance of the model as we explain in the next section. In addition, the lagrangian interpretation of these equations was not studied in those references either.

As for the equations of motion of the fields corresponding to N and M , we obtain from (2.18) and (2.19) that

$$\partial_- (\partial_+ M M^{-1})_{>-l} = 0 \quad \partial_+ (N^{-1} \partial_- N)_{<l} = 0 \quad (2.28)$$

However, one can get more information on the dynamics of such fields from the constraints (2.22) and (2.23). We write

$$\begin{aligned} N &\equiv \exp(\sum_{s>0} \zeta_s) \quad , \quad \zeta_s \in \hat{\mathcal{G}}_s \\ M &\equiv \exp(\sum_{s>0} \xi_{-s}) \quad , \quad \xi_{-s} \in \hat{\mathcal{G}}_{-s} \end{aligned} \quad (2.29)$$

and we label the negative grades in a ordered way as $-s_1 > -s_2 > -s_3 > \dots$. Using the fact that $\partial e^T e^{-T} = \partial T + \frac{1}{2!}[T, \partial T] + \frac{1}{3!}[T[T, \partial T]] + \dots$, one observes that the term in $\partial_- M M^{-1}$

whose grade is maximal is just $\partial_- \xi_{-s_1}$. Therefore, if $s_1 < l$, (2.25) requires that $\partial_- \xi_{-s_1} = 0$. Using this result, the term whose grade is next to $-s_1$ in $\partial_- MM^{-1}$ is $\partial_- \xi_{-s_2}$ and, therefore, if $s_2 < l$, (2.25) implies $\partial_- \xi_{-s_2} = 0$. So, following such reasoning, one gets

$$\partial_- \xi_{-s} = 0 \quad \text{for } s < l \quad (2.30)$$

Analogously, for the fields in N one gets

$$\partial_+ \zeta_s = 0 \quad \text{for } s < l \quad (2.31)$$

Therefore the extra fields in N and M are chiral and decouple from the B fields. Obviously, when l corresponds to the lowest positive grade, the reduced model contains only B and such chiral fields are absent.

From (2.20)-(2.21) and (2.22)-(2.23) one obtains

$$\begin{aligned} \partial_- \hat{g} \hat{g}^{-1} &= N \left(\Lambda_{-l} + \partial_- B B^{-1} \right) N^{-1} + \partial_- N N^{-1} \\ &\equiv \Lambda_{-l} + \sum_{s > -l} j_L^{(s)} \end{aligned} \quad (2.32)$$

$$\begin{aligned} \hat{g}^{-1} \partial_+ \hat{g} &= M^{-1} \left(\Lambda_l + B^{-1} \partial_+ B \right) M + M^{-1} \partial_+ M \\ &\equiv \Lambda_l + \sum_{s < l} j_R^{(s)} \end{aligned} \quad (2.33)$$

where the index s denotes the grade of the current component.

Therefore, in terms of the currents, the constraints (2.22)-(2.23) are

$$\left(\partial_- \hat{g} \hat{g}^{-1} \right)_{-l < s < 0} = \left(N \Lambda_{-l} N^{-1} \right)_{-l < s < 0} \quad (2.34)$$

$$\left(\partial_- \hat{g} \hat{g}^{-1} \right)_{-l} = \Lambda_{-l} \quad (2.35)$$

$$\left(\partial_- \hat{g} \hat{g}^{-1} \right)_{< -l} = 0 \quad (2.36)$$

and

$$\left(\hat{g}^{-1} \partial_+ \hat{g} \right)_{> l} = 0 \quad (2.37)$$

$$\left(\hat{g}^{-1} \partial_+ \hat{g} \right)_l = \Lambda_l \quad (2.38)$$

$$\left(\hat{g}^{-1} \partial_+ \hat{g} \right)_{0 < s < l} = \left(M^{-1} \Lambda_l M \right)_{0 < s < l} \quad (2.39)$$

When l is not the lowest positive grade in (2.13), these constraints are not only first class, but they also include second class constraints.

3 Conformal Invariance of the G-CAT models

The two-loop WZNW model (2.1) is a conformally invariant theory and the corresponding Virasoro generators are given by the Sugawara construction. The procedure to show that

the G-CAT models are also conformal invariant is the same for both chiralities, and, here, we consider only the right moving component of the stress tensor. From now on, we shall restrict ourselves to the models defined in terms of integral gradations (see appendices B and C).

Using the Sugawara construction we can obtain two Virasoro generators from the currents of the two-loop Kac-Moody algebra (see (2.6)-(2.9) and (2.10))

$$T^{(1)}(x) = \frac{1}{2k} \sum_{a,b=1}^{\dim \mathcal{G}} \sum_{n=-\infty}^{\infty} g^{ab} J_{R,a}^n(x) J_{R,b}^{-n}(x) \quad (3.1)$$

$$T^{(2)}(x) = \frac{1}{k} J_R^{\mathcal{D}}(x) J_R^{\mathcal{C}}(x) \quad (3.2)$$

They both satisfy a centreless Virasoro algebra

$$[T^{(i)}(x), T^{(i)}(y)] = 2 T^{(i)}(y) \partial_x \delta(x-y) - \partial_y (T^{(i)}(y)) \delta(x-y); \quad \text{for } i = 1, 2 \quad (3.3)$$

and they commute

$$[T^{(1)}(x), T^{(2)}(y)] = 0 \quad (3.4)$$

These Virasoro generators induce the following transformation of the currents:

$$\begin{aligned} [T^{(1)}(x), J_{R,a}^n(y)] &= J_{R,a}^n(y) \delta'(x-y) - \partial_y (J_{R,a}^n(y)) \delta(x-y) \\ &\quad - \frac{n}{k} J_{R,a}^n(y) J_R^{\mathcal{C}}(y) \delta(x-y) \end{aligned} \quad (3.5)$$

$$[T^{(1)}(x), J_R^{\mathcal{D},\mathcal{C}}(y)] = 0 \quad (3.6)$$

and

$$\begin{aligned} [T^{(2)}(x), J_{R,a}^n(y)] &= \frac{n}{k} J_{R,a}^n(y) J_R^{\mathcal{C}}(y) \delta(x-y) \\ [T^{(2)}(x), J_R^{\mathcal{D},\mathcal{C}}(y)] &= J_R^{\mathcal{D},\mathcal{C}}(y) \delta'(x-y) - \partial_y (J_R^{\mathcal{D},\mathcal{C}}(y)) \delta(x-y) \end{aligned} \quad (3.7)$$

Therefore all the currents transform as primary fields of conformal weight 1, under the sum of the two Virasoro generators

$$T(x) \equiv T^{(1)}(x) + T^{(2)}(x) \quad (3.8)$$

and so

$$[T(x), \mathcal{J}(y)] = \mathcal{J}(y) \partial_x \delta(x-y) - \partial_y (\mathcal{J}(y)) \delta(x-y) \quad (3.9)$$

where $\mathcal{J}(y)$ stands for any of the currents. Obviously $T(x)$ also satisfies a centreless Virasoro algebra

$$[T(x), T(y)] = 2T(y) \partial_x \delta(x-y) - \partial_y (T(y)) \delta(x-y) \quad (3.10)$$

In (2.38) the currents in the direction of the generators of grade l were set to a constant. This breaks the conformal invariance associated to $T(x)$ since such currents are not scalars. However, we can modify $T(x)$ to obtain a new Virasoro generator under which those currents

are scalars. If the gradation (2.13) is realized by a grading operator Q_s , we can use the component of the current corresponding to Q_s , *i.e.*,

$$J_R^{Q_s}(x) = \text{Tr}(Q_s \mathcal{J}_R(x)), \quad (3.11)$$

to modify $T(x)$; then, from (2.10) and (2.12), the right currents in the direction of the generators of grade l have grade $-l$ with respect to $J_R^{Q_s}$

$$[J_R^{Q_s}(x), J_R^{(-l)}(y)] = -l J_R^{(-l)}(y) \delta(x-y). \quad (3.12)$$

Therefore, we introduce the improved stress tensor as

$$L(x) \equiv T(x) + \frac{1}{l} \partial_x J_R^{Q_s}(x) \quad (3.13)$$

under which the currents set to constant are scalars.

The improved stress tensor (3.13) satisfies

$$[L(x), L(y)] = 2L(y) \partial_x \delta(x-y) - \partial_y L(y) \delta(x-y), \quad (3.14)$$

which is a centreless Virasoro algebra too; notice that the Virasoro algebra generated by $L(x)$ might have a central extension proportional to $\text{Tr}(Q_s^2)$, but, for the particular choice (C.1), it vanishes. With respect to $L(x)$, the components of the current $\mathcal{J}_R(x)$ whose grade is $j \in \mathbb{Z}$ transform as primary fields of conformal weight $1 - j/l$, with the exception of the component $\text{Tr}(C \mathcal{J}_R(x))$ whose transformation is

$$\begin{aligned} [L(x), \text{Tr}(C \mathcal{J}_R(y))] &= \text{Tr}(C \mathcal{J}_R(y)) \partial_x \delta(x-y) \\ &\quad - \partial_y \left(\text{Tr}(C \mathcal{J}_R(y)) \right) \delta(x-y) + \frac{k N_s}{l} \partial_x^2 \delta(x-y). \end{aligned} \quad (3.15)$$

For the other chirality a similar procedure applies, thus establishing the conformal invariance of the G-CAT models.

According to eq.(C.10), we parameterize the field B as

$$B \equiv B_0 e^{\nu C + \eta Q_s} \quad (3.16)$$

where B_0 denotes an element of the subgroup formed by exponentiating the elements of the subalgebra $\hat{\mathcal{G}}_0^*$ defined in appendix C. From (2.26), we get

$$\partial_+ (\partial_- B_0 B_0^{-1}) + \partial_+ \partial_- \nu C + \partial_+ \partial_- \eta Q_s = e^{l\eta} [\Lambda_{-l}, B_0 \Lambda_l B_0^{-1}] \quad (3.17)$$

The conformal invariance of the G-CAT model can be made explicit by verifying that the above equation is invariant under the conformal transformations

$$x_+ \rightarrow \tilde{x}_+ = f(x_+); \quad x_- \rightarrow \tilde{x}_- = g(x_-) \quad (3.18)$$

if the fields transform as

$$\begin{aligned} B_0(x_+, x_-) &\rightarrow \tilde{B}_0(\tilde{x}_+, \tilde{x}_-) = B_0(x_+, x_-) \\ e^{-\nu(x_+, x_-)} &\rightarrow e^{-\tilde{\nu}(\tilde{x}_+, \tilde{x}_-)} = (f'(x_+))^\delta (g'(x_-))^{\bar{\delta}} e^{-\nu(x_+, x_-)} \\ e^{-\eta(x_+, x_-)} &\rightarrow e^{-\tilde{\eta}(\tilde{x}_+, \tilde{x}_-)} = (f'(x_+))^{1/l} (g'(x_-))^{1/l} e^{-\eta(x_+, x_-)} \end{aligned} \quad (3.19)$$

where δ and $\bar{\delta}$ are arbitrary. Notice that $\text{Tr} \left(C \mathcal{J}_{R/L}(x_{\pm}) \right) = k N_s \partial_{\pm} \eta(x_+, x_-)$, and that the transformation expressed by eq.(3.19) is in agreement with eq.(3.15).

The grading operator Q_s has a component in the direction of D that cannot be the result of any commutator, *i.e.*, $D, Q_s \notin [\hat{\mathcal{G}}, \hat{\mathcal{G}}]$; then, it follows from (3.17) that η is actually a free field

$$\partial_+ \partial_- \eta = 0 \quad (3.20)$$

Therefore the solutions for η are of the form

$$\eta(x_+, x_-) = \eta_+(x_+) + \eta_-(x_-), \quad (3.21)$$

and, for every regular solution, the term $e^{l\eta}$ can be eliminated by a coordinate transformation, like in the usual CAT models [14]. In fact, for regular solutions, we can define the coordinate transformation

$$x_+ \rightarrow \tilde{x}_+ \equiv \int^{x_+} dy_+ e^{l\eta_+} \quad x_- \rightarrow \tilde{x}_- \equiv \int^{x_-} dy_- e^{l\eta_-} \quad (3.22)$$

and so

$$\partial_+ (A \partial_- B) \rightarrow e^{l(\eta_+ + \eta_-)} \tilde{\partial}_+ (A \tilde{\partial}_- B) \quad (3.23)$$

Hence, when the equation of motion for η holds, (3.17) becomes

$$\tilde{\partial}_+ (\tilde{\partial}_- B_0 B_0^{-1}) + \tilde{\partial}_+ \tilde{\partial}_- \nu C = [\Lambda_{-l}, B_0 \Lambda_l B_0^{-1}] \quad (3.24)$$

We will refer to such theory, containing only the fields B_0 and ν , as the *Generalized non-abelian affine Toda models* (G-AT). Let us point out that, in (3.24), the dynamics of the field B_0 is actually independent of the field ν , while the equation of motion of ν is, see (C.8)-(C.9),

$$\tilde{\partial}_+ \tilde{\partial}_- \nu = \frac{1}{N_s} \text{Tr} \left(Q_s [\Lambda_{-l}, B_0 \Lambda_l B_0^{-1}] \right) = -\frac{l}{N_s} \text{Tr} \left(\Lambda_l B_0^{-1} \Lambda_{-l} B_0 \right). \quad (3.25)$$

It is also worth noticing that the G-AT model is already invariant under the scale transformation $x_{\pm} \rightarrow \lambda^{\pm 1} x_{\pm}$ (which is, in fact, a Lorentz transformation), and that the field η actually plays the role of a connection to implement conformal invariance.

4 The Leznov-Saveliev solution

In this section, we use the method of Leznov and Saveliev [3, 15] to obtain the general solution for the G-CAT models. This method has been discussed quite extensively in the context of Toda models and we present it here in some detail because, when applied to the G-CAT model, it has some features not common to the other ones. Specifically, when l , the grade of the currents set to a constant value (see (2.35)-(2.38)), is not the lowest non-vanishing grade, the solution has a number of chiral parameters greater than twice the number of fields. In addition, we will use the results of this section to relate some special Leznov-Saveliev solutions to the dressing transformations and τ -functions discussed in sections 7 and 8.

The equations of motion (2.26) (or equivalently (2.27)) can be put in the form of a zero-curvature condition

$$[\partial_+ + A_+, \partial_- + A_-] = 0 \quad (4.1)$$

with the gauge potentials being given by

$$\begin{aligned} A_+ &= -B\Lambda_l B^{-1} \\ A_- &= -\partial_- B B^{-1} + \Lambda_{-l} \end{aligned} \quad (4.2)$$

Notice that the zero-curvature condition (4.1) is equivalent to the equations of motion for the B fields only. It does not involve the equations for the chiral fields ξ_{-s} and ζ_s ($1 < s < l$) given by (2.30)-(2.31).

Since the gauge potentials (4.2) have to satisfy (4.1), they must be of the “pure gauge” form, *i.e.*,

$$A_{\pm} = -\partial_{\pm} g g^{-1} \quad (4.3)$$

where g is an exponentiation of the Kac-Moody algebra generators (2.2)-(2.3). The element g is obtained by integrating (4.3) a la Dyson, *i.e.*, $g(x_+, x_-) = g(0) P \exp \left(\int^{(x_+, x_-)} dy^{\mu} A_{\mu} \right)$. However, since the field strength associated to A_{μ} vanishes, it follows from Stoke’s theorem that the integral is path independent. In general, due to the form of A_{μ} expressed by eq.(4.2), the integration becomes easier by choosing two special paths. Here, we use an algebraic procedure to integrate (4.3), and we start by writing g in terms of two different group elements $g_1, g_2 \in G$ as

$$g \equiv g_1 \equiv B g_2, \quad (4.4)$$

which are related to the two previously mentioned special paths. Then, we get

$$\begin{aligned} \partial_{\pm} g g^{-1} &= \partial_{\pm} g_1 g_1^{-1} \\ \partial_{\pm} g g^{-1} &= \partial_{\pm} B B^{-1} + B \partial_{\pm} g_2 g_2^{-1} B^{-1} \end{aligned} \quad (4.5)$$

Therefore from (4.2) and (4.3)

$$\partial_+ g_1 g_1^{-1} = B \Lambda_l B^{-1} \quad (4.6)$$

$$\partial_- g_1 g_1^{-1} = \partial_- B B^{-1} - \Lambda_{-l} \quad (4.7)$$

$$\partial_+ g_2 g_2^{-1} = -B^{-1} \partial_+ B + \Lambda_l \quad (4.8)$$

$$\partial_- g_2 g_2^{-1} = -B^{-1} \Lambda_{-l} B \quad (4.9)$$

We now consider the states of a highest weight representation of the Kac-Moody algebra $\hat{\mathcal{G}}$ which are annihilated by the positive grade generators, *i.e.*,

$$T | \mu \rangle = 0; \quad \text{for } T \in \hat{\mathcal{G}}_+ \quad (4.10)$$

In addition, we require that

$$\langle \mu | T = 0; \quad \text{for } T \in \hat{\mathcal{G}}_- \quad (4.11)$$

which is a consequence of (4.10) if $\hat{\mathcal{G}}_-$ is related to $\hat{\mathcal{G}}_+$ by conjugation. The set of such highest weight states constitutes a representation of the finite subalgebra $\hat{\mathcal{G}}_0$, since if $| \mu \rangle$ satisfies

(4.10) so does $\hat{\mathcal{G}}_0 | \mu \rangle$; we will assume that this representation of $\hat{\mathcal{G}}_0$ is faithful, which is not true for the integrable highest weight representations mentioned in appendix B.

From (4.6), (4.9), (4.10) and (4.11), one then gets

$$\begin{aligned} \partial_+ g_1 g_1^{-1} | \mu \rangle &= -g_1 \partial_+ g_1^{-1} | \mu \rangle = B \Lambda_l B^{-1} | \mu \rangle = 0 \\ \langle \mu | \partial_- g_2 g_2^{-1} &= -\langle \mu | B^{-1} \Lambda_{-l} B = 0 \end{aligned} \quad (4.12)$$

and so the states $g_1^{-1} | \mu \rangle$ and $\langle \mu | g_2$ are chiral

$$\partial_+ g_1^{-1} | \mu \rangle = 0; \quad \langle \mu | \partial_- g_2 = 0 \quad (4.13)$$

Now, from (4.4),

$$\langle \mu' | B^{-1} | \mu \rangle = \langle \mu' | g_2 g_1^{-1} | \mu \rangle \quad (4.14)$$

and so, the matrix elements of B^{-1} are obtained by contraction of two chiral vectors.

Next, let us use the Gauss decomposition to write

$$g_1 = \mathcal{N} B_- M_-; \quad g_2 = \mathcal{M} B_+ N_+ \quad (4.15)$$

where

$$B_{\pm} \in \exp(\hat{\mathcal{G}}_0), \quad \mathcal{N}, N_+ \in \exp(\hat{\mathcal{G}}_+), \quad \mathcal{M}, M_- \in \exp(\hat{\mathcal{G}}_-) \quad (4.16)$$

Substituting (4.15) into (4.6), one gets

$$\mathcal{N}^{-1} B \Lambda_l B^{-1} \mathcal{N} = \mathcal{N}^{-1} \partial_+ \mathcal{N} + \partial_+ B_- B_-^{-1} + B_- \partial_+ M_- M_-^{-1} B_-^{-1} \quad (4.17)$$

and, consequently,

$$\partial_+ B_- B_-^{-1} = 0 \quad (4.18)$$

$$\partial_+ M_- M_-^{-1} = 0 \quad (4.19)$$

$$\partial_+ \mathcal{N} \mathcal{N}^{-1} = B \Lambda_l B^{-1} \quad (4.20)$$

Analogously, substituting (4.15) into (4.9) one gets

$$-\mathcal{M}^{-1} B^{-1} \Lambda_{-l} B \mathcal{M} = \mathcal{M}^{-1} \partial_- \mathcal{M} + \partial_- B_+ B_+^{-1} + B_+ \partial_- N_+ N_+^{-1} B_+^{-1} \quad (4.21)$$

and, so,

$$\partial_- B_+ B_+^{-1} = 0 \quad (4.22)$$

$$\partial_- N_+ N_+^{-1} = 0 \quad (4.23)$$

$$\partial_- \mathcal{M} \mathcal{M}^{-1} = -B^{-1} \Lambda_{-l} B \quad (4.24)$$

Now, from (4.7),

$$\mathcal{N}^{-1} (\partial_- B B^{-1} - \Lambda_{-l}) \mathcal{N} = \mathcal{N}^{-1} \partial_- \mathcal{N} + \partial_- B_- B_-^{-1} + B_- \partial_- M_- M_-^{-1} B_-^{-1} \quad (4.25)$$

therefore,

$$\partial_- M_- M_-^{-1} = -B_-^{-1} (\mathcal{N}^{-1} \Lambda_{-l} \mathcal{N})_{<0} B_- \quad (4.26)$$

and, from (4.8),

$$\mathcal{M}^{-1} \left(-B^{-1} \partial_+ B + \Lambda_l \right) \mathcal{M} = \mathcal{M}^{-1} \partial_+ \mathcal{M} + \partial_+ B_+ B_+^{-1} + B_+ \partial_+ N_+ N_+^{-1} B_+^{-1} \quad (4.27)$$

which implies

$$\partial_+ N_+ N_+^{-1} = B_+^{-1} \left(\mathcal{M}^{-1} \Lambda_l \mathcal{M} \right)_{>0} B_+ \quad (4.28)$$

Finally, if we write

$$\mathcal{N} = \exp \left(\sum_{s>0} \chi_s^- \right); \quad \mathcal{M} = \exp \left(\sum_{s>0} \chi_{-s}^+ \right) \quad (4.29)$$

with $\chi_s^\pm \in \hat{\mathcal{G}}_s$, we conclude from (4.20) and (4.24) that

$$\begin{aligned} \partial_+ \chi_s^- &= 0; & \text{for } 0 < s < l \\ \partial_- \chi_{-s}^+ &= 0; & \text{for } 0 < s < l, \end{aligned} \quad (4.30)$$

by using arguments similar to those leading to (2.30) and (2.31); these chiral quantities are the only parameters of \mathcal{N} and \mathcal{M} contributing in (4.26) and (4.28) respectively.

Therefore, from (4.14) and the considerations above we conclude that the general solution for the G-CAT model is given by

$$\langle \mu' | B^{-1}(x_+, x_-) | \mu \rangle = \langle \mu' | B_+(x_+) N_+(x_+) M_-^{-1}(x_-) B_-^{-1}(x_-) | \mu \rangle \quad (4.31)$$

where the quantities $N_+(x_+)$ and $M_-(x_-)$ are determined from (4.28) and (4.26) respectively. The group elements $B_+(x_+)$ and $B_-(x_-)$ together with the chiral quantities $\chi_s^-(x_-)$ and $\chi_{-s}^+(x_+)$ for $0 < s < l$, parameterize the general solution (4.31). When l is the lowest positive grade of $\hat{\mathcal{G}}$ in the gradation, the parameters χ_s^\pm are absent.

Notice that, in order to reconstruct the group element B in (4.31), one has to choose a representation and a suitable number of states $|\mu\rangle$. A priori, the above method does not give any criterium for that choice. In section 8 we will see that, in the τ -function formalism, one needs just the highest weight state of a particular integrable representation of the Kac-Moody algebra $\hat{\mathcal{G}}$.

5 The symmetries and vacua of the G-CAT models

The equations of motion (2.26), or equivalently (2.27), can be obtained from the action

$$\begin{aligned} S &= \kappa \left(-\frac{1}{8} \int d^2 x \operatorname{Tr} \left(\partial_\mu B \partial^\mu B^{-1} \right) + \frac{1}{12} \int d^3 y \epsilon^{ijk} \operatorname{Tr} \left(B^{-1} \partial_i B B^{-1} \partial_j B B^{-1} \partial_k B \right) \right. \\ &\quad \left. + \int d^2 x \operatorname{Tr} \left(\Lambda_l B^{-1} \Lambda_{-l} B \right) \right), \end{aligned} \quad (5.1)$$

i.e., the WZNW action for the field $B \in \exp(\hat{\mathcal{G}}_0)$ plus a potential, where, since $\Lambda_{\pm l} \notin \hat{\mathcal{G}}_0$, Tr has to be the trace form of the Kac-Moody algebra $\hat{\mathcal{G}}$. Nevertheless, the calculation of almost

all the terms in (5.1) requires just the restriction of the trace form of $\hat{\mathcal{G}}$ to $\hat{\mathcal{G}}_0$. Actually, the first two terms of (5.1) involve generators of $\hat{\mathcal{G}}_0$ only, and, in the last term, the trace form can be evaluated on the subalgebra $\hat{\mathcal{G}}_0$ except for a constant term. Indeed, writing $B = e^T$ with $T \in \hat{\mathcal{G}}_0$, one has $\text{Tr}(\Lambda_l B^{-1} \Lambda_{-l} B) = \text{Tr}(\Lambda_l \Lambda_{-l}) - \text{Tr}([\Lambda_{-l}, \Lambda_l] T) + \frac{1}{2} \text{Tr}([[[T, \Lambda_{-l}], \Lambda_l] T) + \dots$. In the action (5.1), notice that κ is a coupling constant which plays no role in the classical theory, where it can be considered just as an overall factor.

The relation of the action (5.1) with the affine Toda models can be exhibited by using the parameterization (C.10) such that

$$B = B_0 e^{\nu C + \eta Q_s}, \quad B_0 \in \exp(\hat{\mathcal{G}}_0^*); \quad (5.2)$$

so, using eqs.(C.11)-(C.13),

$$\begin{aligned} S = & \kappa \left(-\frac{1}{8} \int d^2 x \text{Tr}(\partial_\mu B_0 \partial^\mu B_0^{-1}) + \frac{1}{12} \int d^3 y \epsilon^{ijk} \text{Tr}(B_0^{-1} \partial_i B_0 B_0^{-1} \partial_j B_0 B_0^{-1} \partial_k B_0) \right. \\ & \left. + \int d^2 x e^{l\eta} \text{Tr}(\Lambda_l B_0^{-1} \Lambda_{-l} B_0) + \frac{1}{4} N_s \int d^2 x \partial_\mu \nu \partial^\mu \eta \right). \end{aligned} \quad (5.3)$$

Then, the action of the G-CAT model is the WZNW action for the field $B_0 \in \exp(\hat{\mathcal{G}}_0^*)$, plus a potential involving the field B_0 coupled to η , plus, finally, a kinetic term for the two fields ν and η . Consequently, when $\eta = 0$ (or constant, in general) the action (5.3) is precisely the action of the generalized non-abelian affine Toda model corresponding to the field B_0 .

Eq.(5.3) also shows that one should not expect to obtain always different models for different values of l . First, notice that the explicit dependence on l in $e^{l\eta}$ can be eliminated by $\eta \rightarrow \eta/l$ and $\nu \rightarrow \nu l$. Second, it follows from eqs.(C.1)-(C.3) that

$$\hat{\mathcal{G}} = \mathbb{R} C \oplus \mathbb{R} D \bigoplus_{j \in E} \hat{\mathcal{G}}_j, \quad (5.4)$$

where $E = I + \mathbb{Z} N_s$, I is a set of integers ≥ 0 and $< N_s$, and $\hat{\mathcal{G}}_{j+m N_s}$ is isomorphic to $\hat{\mathcal{G}}_j$ for all $m \in \mathbb{Z}$. Through this isomorphism, for each $u \in \hat{\mathcal{G}}_j$ and $v \in \hat{\mathcal{G}}_{-j}$ such that $\text{Tr}(u v) \neq 0$ there exists $\tilde{u} \in \hat{\mathcal{G}}_{j+m N_s}$ and $\tilde{v} \in \hat{\mathcal{G}}_{-j-m N_s}$, for all $m \in \mathbb{Z} \geq 0$, such that $\text{Tr}(u v) = \text{Tr}(\tilde{u} \tilde{v})$. All this shows that the only *a priori* different models correspond to the different choices of $\Lambda_{\pm l}$ with $0 < l \leq N_s$.

The symmetry group of the G-CAT model is the subgroup of the symmetry group of the WZNW model (2.5) which leaves the potential term invariant. Specifically, (5.1) is invariant under the transformations

$$B(x_+, x_-) \rightarrow h_L(x_-) B(x_+, x_-) \quad , \quad B(x_+, x_-) \rightarrow B(x_+, x_-) h_R(x_+) \quad (5.5)$$

where $h_{L/R}$ are elements of certain subgroups $\mathcal{H}_{L/R} \subset \exp(\hat{\mathcal{G}}_0)$ satisfying

$$h_L^{-1} \Lambda_{-l} h_L = \Lambda_{-l} \quad , \quad h_R \Lambda_l h_R^{-1} = \Lambda_l \quad (5.6)$$

Depending upon the choice of $\Lambda_{\pm l}$, these subgroups may vary considerably, and, in fact, the left and right subgroups might not be isomorphic. The Noether currents associated to such symmetries are

$$J(T_L) = -\text{Tr}(T_L \partial_- B B^{-1}) ; \quad J(T_R) = \text{Tr}(T_R B^{-1} \partial_+ B) \quad (5.7)$$

where

$$T_{L/R} \in \text{Ker}(\text{ad } \Lambda_{\mp l}) \cap \hat{\mathcal{G}}_0, \quad (5.8)$$

i.e.,

$$[T_L, \Lambda_{-l}] = 0, \quad [T_R, \Lambda_l] = 0, \quad \text{and} \quad \exp(T_{L/R}) \in \mathcal{H}_{L/R}. \quad (5.9)$$

Indeed, from (2.26) and (2.27) we have

$$\partial_+ J(T_L) = 0; \quad \partial_- J(T_R) = 0 \quad (5.10)$$

The question of the vacua of these models is quite delicate because the sign of the kinetic energy depends on the properties of the trace form Tr . Actually, we will be interested in the solitons of the massive G-AT models (3.24), which correspond to the spontaneous breakdown of the conformal symmetry by the choice $\eta = \eta_0 = \text{constant}$ (see (6.10)). Then, the only field will be B_0 , and eq.(5.3) shows that the sign of the kinetic term depends only on the restriction of the trace form to $\hat{\mathcal{G}}_0^*$ or, equivalently, on the properties of the subgroup $\exp(\hat{\mathcal{G}}_0^*)$ regarding its compactness. Even though affine Toda models based on non-compact groups have been considered in relation to two-dimensional black holes [4], we will be mainly interested in the case when the group is compact and the kinetic energy of B_0 is positive. This can be achieved by choosing the compact real form of the Lie algebra $\hat{\mathcal{G}}_0^*$, which is always possible because of the following reason. Let us recall that $\hat{\mathcal{G}}_0^*$ is isomorphic to the affine subalgebra of $\hat{\mathcal{G}}$ generated by $\{H_1^0, \dots, H_r^0\}$ ⁴, and $\{E_\alpha^n\}$ with $n N_s + \alpha \cdot H_s = 0$ —in appendix C, we show that the only possible values of n are either 0, if $s_0 \neq 0$, or $0, \pm 1$, if $s_0 = 0$ —. Then, if $E_\alpha^n \in \hat{\mathcal{G}}_0^*$ for some α and n , $E_{-\alpha}^{-n} \in \hat{\mathcal{G}}_0^*$ too, and we can always choose the real form of $\hat{\mathcal{G}}_0^*$ such that its generators are H_1^0, \dots, H_r^0 , $(E_\alpha^n + E_{-\alpha}^{-n})$ and $i(E_\alpha^n - E_{-\alpha}^{-n})$, where $n N_s + \alpha \cdot H_s = 0$. With this choice, eqs.(B.7)-(B.9) can be used to show that the trace form of $\hat{\mathcal{G}}$ restricted to $\hat{\mathcal{G}}_0^*$ is positive definite; this is equivalent to say that the subgroup $\exp(\hat{\mathcal{G}}_0^*)$ is compact. In the previous discussion, we have ignored the field ν because it does not contribute to the kinetic energy of the G-AT model, as can be checked in (5.3). Thus, for $\eta = \eta_0 = \text{constant}$, the action (5.3) has a positive definite kinetic energy for the particular choice of the real form of $\hat{\mathcal{G}}_0^*$.

In addition, with this choice, B_0 has to be unitary, which ensures the reality of the first two terms of (5.3) that specify a particular WZNW model. In contrast, the reality of the potential depends on the hermiticity properties of $\Lambda_{\pm l}$, which strongly constrains the possible choices of $\Lambda_{\pm l}$ [29]. However, as we discuss in sections 6 and 9, the masses of the fundamental particles and solitons of these models depend only upon the eigenvalues of certain elements $E_{\pm l} \in \hat{\mathcal{G}}_{\pm l}$ defined in (5.11), and those masses are real even for a wide class of models whose potential is complex.

In order to have static solutions for the field B_0 , the right hand side of (3.17) should vanish or have components in the direction of the central term only, *i.e.*, we will be interested in those cases where there exists a constant group element $b_0 \in \hat{\mathcal{G}}_0$ such that

$$E_l \equiv b_0 \Lambda_l b_0^{-1} \quad \text{and} \quad E_{-l} \equiv \Lambda_{-l} \quad (5.11)$$

⁴Actually, the generators are $\{\tilde{H}_1^0, \dots, \tilde{H}_r^0\}$, but, because of equation (C.7), the H_a 's are equivalent to the \tilde{H}_a 's as generators of the subalgebra $\hat{\mathcal{G}}_0^*$.

satisfy

$$[E_l, E_{-l}] = \beta C, \quad (5.12)$$

where

$$\beta = \frac{1}{N_s} \text{Tr}(Q_s[E_l, E_{-l}]) = \frac{l}{N_s} \text{Tr}(E_l E_{-l}). \quad (5.13)$$

Notice that eq.(5.12) would be satisfied if $E_{\pm l}$ are elements of some Heisenberg subalgebra of $\hat{\mathcal{G}}$; this shows that it is possible to associate G-CAT models that admit static solutions to the elements of the different Heisenberg subalgebras of $\hat{\mathcal{G}}$ whose classification is available [30]. Then, for $\eta = 0$, the corresponding solution of (3.17) is

$$B_0^{\text{vac}} = b_0 \quad \text{and} \quad \nu_0 = -\beta x_+ x_-. \quad (5.14)$$

Obviously, if $\beta \neq 0$, the vacuum solution for the field in the direction of the central term C is not actually static; in fact, this is a generalization of what occurs in the abelian CAT models [9].

Let us introduce a group element $b^* \in \hat{\mathcal{G}}_0^*$ such that (see eq. (3.16))

$$B = B_0 e^{\nu C + \eta Q_s} = b b_0 e^{\nu_0 C} e^{\eta Q_s} = b^* e^{\nu C} b_0 e^{\nu_0 C} e^{\eta Q_s}; \quad (5.15)$$

so, the vacuum solution is $b^* = 1$, $\nu = 0$, and $\eta = 0$. Therefore, the equations of motion (2.26) read

$$\partial_+ (\partial_- b^* b^{*-1}) + (\partial_+ \partial_- \nu - \beta) C + \partial_+ \partial_- \eta Q_s = e^{l\eta} [E_{-l}, b^* E_l b^{*-1}]; \quad (5.16)$$

obviously, according to (3.20), η is a free field. In terms of the field b^* , the transformations (5.5) read

$$b^* e^{\nu C} \rightarrow \tilde{h}_L(x_-) b^* e^{\nu C}; \quad b^* e^{\nu C} \rightarrow b^* e^{\nu C} \tilde{h}_R(x_+) \quad (5.17)$$

where

$$\tilde{h}_R(x_+) \equiv b_0 h_R(x_+) b_0^{-1}, \quad \tilde{h}_L(x_+) \equiv h_L(x_+), \quad (5.18)$$

and so from (5.6) and (5.11)

$$\tilde{h}_L(x_-)^{-1} E_{-l} \tilde{h}_L(x_-) = E_{-l} \quad \text{and} \quad \tilde{h}_R(x_+) E_l \tilde{h}_R(x_+)^{-1} = E_l. \quad (5.19)$$

In addition to the continuous symmetries expressed by eqs.(5.17)-(5.19), the theory may possess some discrete symmetries. In general, these are generated by group elements which are exponentiations of generators that do not really commute with $E_{\pm l}$, but do leave them invariant when the parameters in the exponentiation assume special values. An interesting class of such symmetries is related to the co-weight lattice of \mathcal{G} . For the integral gradations (B.11), the Cartan subalgebra of $\hat{\mathcal{G}}_0$ is the same as that of $\hat{\mathcal{G}}$. Then, let μ^v be a co-weight of \mathcal{G} , i.e., $\mu^v = \sum_{a=1}^r m_a 2\lambda_a / \alpha_a^2$, with m_a being integers, and λ_a and α_a being the fundamental weights and simple roots of \mathcal{G} , respectively. Any element of the Kac-Moody algebra $\hat{\mathcal{G}}$ has an integer eigenvalue with respect to the elements of the Cartan subalgebra of the form $\mu^v \cdot H^0 + n D$; therefore

$$e^{2\pi i(\mu^v \cdot H^0 + n D)} E_{\pm l} e^{-2\pi i(\mu^v \cdot H^0 + n D)} = E_{\pm l}, \quad \text{for any co-weight } \mu^v \text{ and integer } n \quad (5.20)$$

Obviously, this is a complex transformation; however, since we will be dealing with the compact real form of $\exp(\hat{\mathcal{G}}_0^*)$, it implies a real transformation on the components of the field b , when we parameterize it as in (6.1). Such class of transformations constitute a generalization of what occurs in the abelian affine Toda models, in the pure imaginary coupling constant regime [31, 9, 20]. Those transformations are responsible for the infinitely many degenerate vacua and for the existence of topological solitons. In addition to (5.20), some theories may possess some additional discrete symmetries depending upon the particular form of $E_{\pm l}$. The question of the topological charges associated to such discrete symmetries will be discussed elsewhere.

6 The classical masses of the fundamental particles

The G-CAT models, described by (5.1), are conformally invariant and, therefore, their fundamental particles are massless. However, the G-AT models introduced in (3.24) are the gauge fixed version of the G-CAT models, which are not conformally invariant, and which have massive fundamental particles; in this section, we will calculate their classical masses. In (5.15), we will parameterize the group element $b^* \in \hat{\mathcal{G}}_0^*$ as

$$b^* \equiv e^{iT} \quad (6.1)$$

where

$$T \equiv \varphi^i T_i, \quad i = 1, 2, \dots, \dim \hat{\mathcal{G}}_0^*, \quad (6.2)$$

where $\dim \hat{\mathcal{G}}_0^* = \dim \hat{\mathcal{G}}_0 - 2$, and T_i denote all the generators of $\hat{\mathcal{G}}_0^*$ (see (C.8)); $\{\varphi^i\}$ and ν are fields of the the G-AT model. Then, taking $\eta = 0$, the linear part of the eq.(5.16) gives the Klein-Gordon equations for the G-AT fields

$$\partial_+ \partial_- \nu = 0 \quad (6.3)$$

$$\partial_+ \partial_- \varphi^i T_i + \varphi^i [E_{-l}, [E_l, T_i]] = 0 \quad (6.4)$$

So, the field ν is massless, and, writing (6.4) as (our conventions are $x_{\pm} = x \pm t$, and $\partial^2 = -4\partial_+ \partial_- = \partial_t^2 - \partial_x^2$)

$$(\partial^2 \varphi^j + \varphi^i M_i^{2j}) T_j = 0, \quad (6.5)$$

we get that the mass square matrix for the φ^i fields is given by

$$M_i^{2j} T_j \equiv -4[E_{-l}, [E_l, T_i]] = -4[E_l, [E_{-l}, T_i]], \quad (6.6)$$

where we have used eq. (5.12); therefore

$$\begin{aligned} M_{ij}^2 \equiv M_i^{2k} \text{Tr}(T_k T_j) &= -2(\text{Tr}([E_{-l}, [E_l, T_i]]T_j) + \text{Tr}([E_{-l}, [E_l, T_j]]T_i)) \\ &= -2(\text{Tr}([E_{-l}, T_j][T_i, E_l]) + \text{Tr}([E_{-l}, T_i][T_j, E_l])) \end{aligned} \quad (6.7)$$

If $\tilde{h}_L(x_-)$ and $\tilde{h}_R(x_+)$, defined in (5.19), have a set of common generators, one observes from the second line in (6.7) that the mass matrix is block diagonal with the block corresponding

to this set being zero; so, the particles associated to that set of common generators are massless.

Notice that the diagonalization of the mass matrix (6.6) corresponds to the diagonalization of the action of the operator $[E_{-l}, [E_l, \cdot]]$ on the subalgebra $\hat{\mathcal{G}}_0^*$,

$$[E_{-l}, [E_l, T]] = [E_l, [E_{-l}, T]] = \lambda T ; \quad T \in \hat{\mathcal{G}}_0^* \quad (6.8)$$

Due to (5.12) and to the invariance of the trace form, the subspaces corresponding to different eigenvalues are orthogonal. Therefore, the masses of the fundamental particles are

$$m_\lambda^2 = -4\lambda. \quad (6.9)$$

For the G-CAT models, this constitutes the generalization of the arguments used in the abelian affine Toda models [32, 33]. However, to establish the physical significance of (6.9) in each G-CAT model, one has to ensure that the eigenvalues λ are real and non-positive. Below, we discuss how to calculate the eigenvalues λ in some examples.

When going from the G-CAT to the G-AT model, notice that if we had set the field η to a non-vanishing constant value, namely $\eta = \eta_0$, the commutator term in (5.16) would get multiplied by a factor

$$v_\eta \equiv e^{l\eta_0} \quad (6.10)$$

and all the masses would be multiplied by this factor too, $m_\lambda^2 = -4v_\eta\lambda$. Therefore, the primary field $\phi \equiv e^{l\eta}$ (3.19) actually works like a Higgs field with the mass matrix being proportional to its vacuum expectation value v_η . Consequently, the masses of the G-AT model are generated by the spontaneous breakdown of the conformal invariance of the G-CAT models by the choice of a particular vacuum configuration $\eta = \eta_0 = \text{constant}$. However, we will not carry the constant v_η explicitly because it can be absorbed in the definition of $E_{\pm l}$; so, the mass scale of the G-AT models will be set by the normalization of $E_{\pm l}$.

7 The Dressing Transformations

Consider a generic non-linear system admitting a formulation in terms of a zero-curvature condition

$$[\partial_\mu + A_\mu, \partial_\nu + A_\nu] = 0 \quad (7.1)$$

with the gauge potentials A_μ lying on a Lie algebra $\hat{\mathcal{G}}$ and being functionals of the fields of the model. The dressing transformations [16, 17, 18, 19] are non-local gauge transformations acting on A_μ and leaving its form invariant. Each one of these gauge transformations is performed in two different ways in terms of two different group elements Θ_+ and Θ_- of G (the group whose Lie algebra is $\hat{\mathcal{G}}$)

$$A_\mu \rightarrow A_\mu^g \equiv \Theta_\pm A_\mu \Theta_\pm^{-1} - \partial_\mu \Theta_\pm \Theta_\pm^{-1} \quad (7.2)$$

Since A_μ satisfies (7.1) it has to be of pure gauge form

$$A_\mu = -\partial_\mu T T^{-1} \quad (7.3)$$

and, consequently, $A_\mu^g = -\partial_\mu(\Theta_\pm T)(\Theta_\pm T)^{-1}$. Therefore, there exists a constant group element g such that $\Theta_+ T = \Theta_- T g$ and, so,

$$\Theta_+ T = \Theta_- T g \quad \text{or} \quad \text{equivalently} \quad \Theta_-^{-1} \Theta_+ = T g T^{-1} \quad (7.4)$$

The requirement that the dressing transformation preserves the form of the potential A_μ implies that Θ_\pm must belong to two different subgroups of G determined by the particular form of A_μ in $\hat{\mathcal{G}}$. In this sense, eq. (7.4) corresponds to the factorization of $T g T^{-1}$ into those subgroups. Then, given a solution of the model defined by a group element T and a constant group element g , the dressing transformation can be used to construct another solution defined by $T^g = \Theta_+ T$ (or equivalently $T^g = \Theta_- T$).

We will now use the dressing transformations for the G-CAT models to construct their solutions lying in the orbit of the vacuum. The gauge potentials for the G-CAT models are given by (4.2). Using (5.11) and (5.15) one gets

$$\begin{aligned} A_+ &= -e^{l\eta} b E_l b^{-1} \\ A_- &= -\partial_- b b^{-1} - \partial_- \eta Q_s + E_{-l} + \beta x_+ C. \end{aligned} \quad (7.5)$$

Then, for the vacuum solution, namely $b = b^* = 1$ and $\eta = 0$, one has

$$\begin{aligned} A_+^{(0)} &= -E_l \\ A_-^{(0)} &= E_{-l} + \beta x_+ C \end{aligned} \quad (7.6)$$

which can be written as in (7.3) with the group element

$$T_0 = e^{x_+ E_l} e^{-x_- E_{-l}} \quad (7.7)$$

We now perform a gauge transformation of the form (7.2) that maps (7.6) into (7.5) for $\eta = 0$. Due to the structure of A_\pm , we choose the elements Θ_\pm in (7.2) as

$$\Theta_+ = \Theta_+^{(0)} \Theta_+^>; \quad \Theta_- = \Theta_-^{(0)} \Theta_-^\leq \quad (7.8)$$

with $\Theta_\pm^{(0)}$, $\Theta_+^>$ and Θ_-^\leq belonging to the subgroups of G obtained by exponentiating the subalgebras $\hat{\mathcal{G}}_0$, $\hat{\mathcal{G}}_+$ and $\hat{\mathcal{G}}_-$, defined in (2.15), respectively; consequently, we write

$$\Theta_+^> = \exp\left(\sum_{s>0} t^{(s)}\right); \quad \Theta_-^\leq = \exp\left(\sum_{s>0} t^{(-s)}\right); \quad t^{(\pm s)} \in \hat{\mathcal{G}}_{\pm s} \quad (7.9)$$

Therefore Θ_\pm have to satisfy

$$b E_l b^{-1} = \Theta_\pm E_l \Theta_\pm^{-1} + \partial_+ \Theta_\pm \Theta_\pm^{-1} \quad (7.10)$$

$$E_{-l} - \partial_- b b^{-1} = \Theta_\pm E_{-l} \Theta_\pm^{-1} - \partial_- \Theta_\pm \Theta_\pm^{-1} \quad (7.11)$$

In order to solve these equations one has to split them into the eigensubspaces of the gradation (2.13). We now show how to construct Θ_+ . The calculations for Θ_- are very similar. Due to the structure of Θ_+ the right hand side of (7.10) and (7.11) have grades greater

or equal to 0 and $-l$ respectively. So, using (7.8) one gets that the component of grade zero in (7.10) implies $\partial_+ \Theta_+^{(0)} (\Theta_+^{(0)})^{-1} = 0$ and the component of grade $-l$ of (7.11) implies $E_{-l} = \Theta_+^{(0)} E_{-l} (\Theta_+^{(0)})^{-1}$. Therefore, $\Theta_+^{(0)}$ has to be an element of the subgroup \mathcal{H}_L introduced in (5.5). So, we write

$$\Theta_+^{(0)} \equiv h_L^{-1}(x_-) \quad (7.12)$$

Considering the component of (7.10) with grades $0 < s < l$, one gets

$$\left(\partial_+ \Theta_+^> (\Theta_+^>)^{-1} \right)_{0 < s < l} = 0, \quad (7.13)$$

and from the component of (7.11) of grades $-l < s < 0$ it follows that

$$\left(\Theta_+^> E_{-l} (\Theta_+^>)^{-1} \right)_{-l < s < 0} = 0. \quad (7.14)$$

Therefore, taking into account (7.9), one concludes that

$$t^{(s)} \equiv t^{(s)}(x_-) \in \text{centralizer of } E_{-l}, \text{ for } 0 < s < l \quad (7.15)$$

Now, using (7.12) and (7.15), the component of grade l of eq.(7.10) leads to

$$\partial_+ t^{(l)} = -E_l + h_L(x_-) b E_l b^{-1} h_L^{-1}(x_-) \quad (7.16)$$

As for the component of grade 0 in (7.11) one gets

$$\partial_- b b^{-1} = -h_L^{-1}(x_-) \partial_- h_L(x_-) - h_L^{-1}(x_-) [t^{(l)}, E_{-l}] h_L(x_-) \quad (7.17)$$

Finally, for the components of (7.10) and (7.11) of grades greater than l and 0 respectively, the result is

$$\left(\partial_+ \Theta_+^> (\Theta_+^>)^{-1} + \Theta_+^> E_l (\Theta_+^>)^{-1} \right)_{>l} = 0 \quad (7.18)$$

$$\left(\partial_- \Theta_+^> (\Theta_+^>)^{-1} - \Theta_+^> E_{-l} (\Theta_+^>)^{-1} \right)_{>0} = 0 \quad (7.19)$$

Performing the same calculation for Θ_- , one finds

$$\Theta_-^{(0)} = b \tilde{h}_R(x_+) \quad (7.20)$$

$$t^{(-s)} = t^{(-s)}(x_+) \in \text{centralizer of } E_l, \text{ for } 0 < s < l \quad (7.21)$$

$$\partial_- t^{(-l)} = E_{-l} - \tilde{h}_R(x_+)^{-1} b^{-1} E_{-l} b \tilde{h}_R(x_+) \quad (7.22)$$

$$b^{-1} \partial_+ b = -\partial_+ \tilde{h}_R(x_+) \tilde{h}_R(x_+)^{-1} - \tilde{h}_R(x_+) [t^{(-l)}, E_l] \tilde{h}_R(x_+)^{-1} \quad (7.23)$$

where $\tilde{h}_R(x_+)$ was defined in (5.18), and also

$$\left(\partial_+ \Theta_-^< (\Theta_-^<)^{-1} + \Theta_-^< E_l (\Theta_-^<)^{-1} \right)_{<0} = 0 \quad (7.24)$$

$$\left(\partial_- \Theta_-^< (\Theta_-^<)^{-1} - \Theta_-^< E_{-l} (\Theta_-^<)^{-1} \right)_{<-l} = 0 \quad (7.25)$$

Taking all this into account, from (7.4), one obtains

$$\begin{aligned} (\Theta_-^\leq)^{-1} \left(h_L(x_-) b \tilde{h}_R(x_+) \right)^{-1} \Theta_+^\geq &= T_0 g T_0^{-1} \\ &= e^{x_+ E_l} e^{-x_- E_{-l}} g e^{x_- E_{-l}} e^{-x_+ E_l} \end{aligned} \quad (7.26)$$

This relation has a very powerful meaning and it will be important in understanding the soliton solutions and τ -functions. Once Θ_+^\geq and Θ_-^\leq are determined from the relations given above, it is possible to reconstruct the corresponding solution for the b fields on the orbit of the vacuum for each constant group element g . In (7.26), notice that the gauge symmetries (5.5) of the G-CAT models have been made explicit, and one can easily choose a unique solution in each one of the orbits of those gauge groups.

Since we will use it in section 10, we point out here that the choice

$$t^{(n)} = t^{(-n)} = 0; \quad \text{for } n \text{ not a multiple of } l \quad (7.27)$$

for the quantities introduced in (7.9) is a solution for the dressing problem that corresponds to choosing all the chiral quantities in eqs.(7.15) and (7.21) to be zero. Indeed (7.15) and (7.21) are trivially satisfied and (7.18), (7.19), (7.24) and (7.25) will now have components only in the subspaces with grade which are multiples of l , and so, they can be consistently solved. However, unless some extraordinary cancellation happens, the type of solutions (7.27) imply that the constant group element g in (7.26) can only be an exponentiation of generators with grades which are multiples of l .

7.1 Connection with the Leznov-Saveliev solution

If $|\mu\rangle$ and $|\mu'\rangle$ are states of a given representation of $\hat{\mathcal{G}}$ satisfying (4.10) and (4.11), and after fixing the gauge as $\tilde{h}_R = h_L = 1$, one obtains from (7.26) that

$$\begin{aligned} \langle \mu' | b^{-1} | \mu \rangle &= \langle \mu' | b_0 B^{-1} e^{\nu_0 C} | \mu \rangle \\ &= e^{-\beta x_+ x_- C(\mu)} \langle \mu' | e^{x_+ E_l} g e^{x_- E_{-l}} | \mu \rangle \end{aligned} \quad (7.28)$$

where we have used (5.12), and $C(\mu)$ is the value of the central term in the representation under consideration.

Now, if one chooses the parameters of the Leznov-Saveliev solution (4.31) as

$$B_+ = b_0^{-1} \quad B_- = 1 \quad (7.29)$$

and

$$\chi_s^- = \chi_{-s}^+ = 0; \quad \text{for } 0 < s < l \quad (7.30)$$

then one gets from (4.26) and (4.28) that

$$M_-(x_-) = e^{-x_- E_{-l}} M_-(0); \quad N_+(x_+) = e^{x_+ E_l} N_+(0) \quad (7.31)$$

Substituting that into (4.31), and using (3.16) and (5.15), we get the same as (7.28) with $g \equiv N_+(0)M_-(0)^{-1}$. The factor $e^{\eta Q_s}$ drops out if the state $|\mu\rangle$ has zero Q_s -eigenvalue, and

a factor b_0^{-1} on both sides of the relation can be eliminated before taking the expectation value on the representation states.

Therefore the quite simple choice of parameters (7.29) and (7.30) leads, in the Leznov-Saveliev method, to all the solutions in the orbit, under the dressing transformations, of the vacuum solution. In addition, such choice picks up only one solution in each orbit of the gauge transformations (5.5). Such observation sheds some light on the so called *solitonic specialisation procedure* used in [20] to construct the soliton solutions of the abelian Affine Toda models, and suggested in [21] for the same construction, in the case of a more general class of integrable models.

8 The τ -functions

In relation (7.26) the solution for the field b is found in terms of the quantities $t^{(s)}$. In practice, solving such relations might turn in a quite cumbersome task. However, the quantities $t^{(s)}$ have a very important meaning. They are the embryo of the so called τ -functions.

The τ -functions have a very important role in soliton theory. They are used in the Hirota method to construct exact solutions of soliton equations [34]. In addition, formalisms have been developed to construct soliton equations starting from a given definition of the τ -function [22, 23]. The connection between dressing transformations and τ -functions has been explored in the context of the KdV equation [35] and of the generalized integrable hierarchies of the KdV type [24]. However, in those cases the τ -functions were defined on some special vertex representation of simply laced Kac-Moody algebras. Inspired on the dressing transformations, here, we present a general definition of the τ -function that is not restricted to any particular representation of $\hat{\mathcal{G}}$, and which generalizes the results of [24].

Consider an integral gradation of an affine Kac-Moody algebra $\hat{\mathcal{G}}$ labelled by a vector \mathbf{s} (B.11), and let $|\lambda_{\mathbf{s}}\rangle$ be the highest weight state of an integrable representation of $\hat{\mathcal{G}}$ [27], satisfying the relations (B.17)-(B.21). We define the τ -function $\tau_{\mathbf{s}}$ as the result of the action of the left hand side of (7.26) (or equivalently, its right hand side) on $|\lambda_{\mathbf{s}}\rangle$, but with the gauge symmetry (5.5) fixed as $h_L(x_-) = \tilde{h}_R(x_+) = 1$, i.e.,

$$\begin{aligned} \tau_{\mathbf{s}}(x_+, x_-) &\equiv (\Theta_-^{\leq})^{-1} b^{-1} |\lambda_{\mathbf{s}}\rangle \\ &= e^{x_+ E_l} e^{-x_- E_{-l}} g e^{x_- E_{-l}} |\lambda_{\mathbf{s}}\rangle \end{aligned} \quad (8.1)$$

As far as the relation among the b fields and τ -functions is concerned, the definition above is made on shell. However once such relation is established it can be extended off shell.

The definition (8.1) implies that

$$\partial_+ \tau_{\mathbf{s}} = [E_l, G] |\lambda_{\mathbf{s}}\rangle = E_l \tau_{\mathbf{s}} \quad (8.2)$$

$$\partial_- \tau_{\mathbf{s}} = -[E_{-l}, G] |\lambda_{\mathbf{s}}\rangle \quad (8.3)$$

where we have used (5.12) and denoted

$$G \equiv e^{x_+ E_l} e^{-x_- E_{-l}} g e^{x_- E_{-l}} e^{-x_+ E_l} = T_0 g T_0^{-1} \quad (8.4)$$

Therefore,

$$\partial_+ \partial_- \tau_s = -[E_l, [E_{-l}, G]] | \lambda_s \rangle, \quad (8.5)$$

Since τ_s is a state of the representation, we can split it into eigenstates of the grading operator Q_s given by (B.12)

$$\tau_s = \tau_s^{(0)} + \tau_s^{(-1)} + \tau_s^{(-2)} + \dots \quad (8.6)$$

with

$$Q_s \tau_s^{(n)} = (\eta_s + n) \tau_s^{(n)}, \quad (8.7)$$

where η_s has been defined in (B.20).

Notice that $| \lambda_s \rangle$ is an eigenstate of the subalgebra $\hat{\mathcal{G}}_0$ and, since Θ_-^\leq is an exponentiation of the negative grade operators, we get that

$$\tau_s^{(0)} = b^{-1} | \lambda_s \rangle \sim | \lambda_s \rangle \quad (8.8)$$

Thus, denoting

$$\hat{\tau}_s^{(0)} \equiv \langle \lambda_s | \tau_s^{(0)} = \langle \lambda_s | \tau_s = \langle \lambda_s | b^{-1} | \lambda_s \rangle \quad (8.9)$$

we get

$$\frac{\tau_s}{\hat{\tau}_s^{(0)}} = (\Theta_-^\leq)^{-1} | \lambda_s \rangle. \quad (8.10)$$

This equation is the straightforward generalization of the eq.(5.1) of [24], which, in that reference, establishes the relation between the zero curvature formalism and the τ -functions for a class of generalized integrable hierarchies of partial differential equations; our results open the possibility of generalizing [24] for a wider class of integrable equations. As a consequence of (8.2), notice that the c-number function $\hat{\tau}_s^{(0)}$ satisfies

$$\partial_+ \ln \hat{\tau}_s^{(0)} = \langle \lambda_s | E_l (\Theta_-^\leq)^{-1} | \lambda_s \rangle \quad (8.11)$$

Solutions that travel with constant velocity without dispersion, namely the one-soliton solutions, can be easily constructed as follows. Let F be an eigenvector under the adjoint action of $E_{\pm l}$

$$[E_{\pm l}, F] = \omega_{\pm l} F \quad (8.12)$$

We then choose the constant group element g in (8.1) as

$$g_{\text{sol.}} \equiv e^F \quad (8.13)$$

Therefore

$$\begin{aligned} \tau_s^{1-\text{sol.}}(x_+, x_-) &\equiv e^{x_+ E_l} e^{-x_- E_{-l}} e^F e^{x_- E_{-l}} | \lambda_s \rangle \\ &= \exp \left(e^{\gamma(x-vt)} F \right) | \lambda_s \rangle \end{aligned} \quad (8.14)$$

where $x_{\pm} = x \pm t$, and

$$\gamma = \omega_l - \omega_{-l}; \quad v = -\frac{\omega_l + \omega_{-l}}{\omega_l - \omega_{-l}} \quad (8.15)$$

Nevertheless, if v is going to be the velocity of the soliton, observe that we must have

$$\omega_l \omega_{-l} < 0 \quad (8.16)$$

because if $\omega_{\pm l}$ have the same sign, then $|v| > 1$.

So, for each eigenvector of $E_{\pm l}$ we have a one-soliton solution. The multi-soliton solutions are constructed by choosing g as a product of one-soliton g 's, *i.e.*, $g = e^F e^{F'} e^{F''} \dots$. In fact, this is a generalisation of the ideas used in [20]. However, notice that if we have F and F' , with corresponding eigenvalues $\omega_{\pm l}$ and $\omega'_{\pm l}$, satisfying (8.15) with different γ 's (let's say γ and γ') but the same velocity v , then we have

$$\begin{aligned} \tau_{\mathbf{s}}^{2-\text{sol.}}(x_+, x_-) &\equiv e^{x_+ E_l} e^{-x_- E_{-l}} e^F e^{F'} e^{x_- E_{-l}} | \lambda_{\mathbf{s}} \rangle \\ &= \exp \left(e^{\gamma(x-vt)} F \right) \exp \left(e^{\gamma'(x-vt)} F' \right) | \lambda_{\mathbf{s}} \rangle \end{aligned} \quad (8.17)$$

So, we have a two-soliton solution which can be put at rest in some Lorentz frame. Such solution possesses several properties of the one-soliton solutions. Indeed, as we discuss below, the mass formula for these solutions are obtained using the same techniques as for the one-soliton solutions. In some special cases, even n -soliton solutions of that type can be constructed. In [9] those types of solutions were constructed for the abelian affine Toda using the Hirota's method.

8.1 Hirota's τ -functions

The concept of Hirota's τ -function is a practical one, in the sense that it just provides a method of constructing solutions [34]. Sometimes, the definition of the τ -function can be given more formally depending upon the level of understanding of the structures of the model. However, from the pragmatic point of view of constructing solutions, one can say that it corresponds to a redefinition of the fields of the model such that the equations of motion acquire a form that can be solved exactly by a *formal* perturbation expansion. For some special classes of solutions, the perturbation procedure gives the exact solution because the expansion truncates at a finite order. In the abelian affine Toda models, the Hirota's τ -functions were used to construct the soliton solutions [31, 9]; but their definition was guided basically by the structure of the equations of motion and the results known for the corresponding one-dimensional version of that models.

An important consequence of the results obtained from the dressing transformation method and the definition (8.1) is that, now, we are able to introduce Hirota's τ -functions for any non-abelian affine Toda model. As we will see in the examples, the Hirota's τ -functions for such models correspond to some components of the τ -function $\tau_{\mathbf{s}}$ (8.1) which provide a convenient parameterization of the b fields. However, since the gauge symmetry (5.19) was fixed in the definition (8.1), there will not be Hirota's τ -functions associated to the gauge fixed degrees of freedom. Roughly speaking, the Hirota's τ -functions have the form

$$\tau_v = \langle v | \tau_{\mathbf{s}} \quad (8.18)$$

where a number of states of the representation $|v\rangle$ is suitably chosen such that the τ_v 's parameterize all the fields. In some cases, like the principal gradation discussed in section 10,

a given τ_v may depend on more than one field and so it has to be split into components by considering one parameter subspaces on the orbit defined by τ_s . Nevertheless, it is always clear which components are needed to describe the fields of the model.

The truncation of the Hirota's expansion occurs if, for instance, in the case of the one-soliton solutions, there exists a positive integer N_v such that the generator F , given in (8.12), satisfies

$$\langle v \mid F^n \mid \lambda_s \rangle = 0; \quad \text{for } n > N_v \quad (8.19)$$

Then, from (8.14), the corresponding Hirota's τ -function for the one-soliton solution truncate

$$\tau_v = \tau_v^{(0)} + \tau_v^{(1)} + \tau_v^{(2)} \dots + \tau_v^{(N_v)} \quad (8.20)$$

where

$$\tau_v^{(n)} = \frac{1}{n!} e^{n\gamma(x-vt)} \langle v \mid F^n \mid \lambda_s \rangle \quad (8.21)$$

This generalizes the case of the level-one representations of affine simply-laced algebras where F is just a nilpotent vertex operator.

9 The Soliton masses

The canonical energy-momentum tensor for the G-CAT model action (5.1) is given by

$$\Theta_{\mu\nu} = \kappa \left(-\frac{1}{4} \text{Tr} \left(\partial_\mu B \partial_\nu B^{-1} \right) + \frac{1}{8} g_{\mu\nu} \text{Tr} \left(\partial_\rho B \partial^\rho B^{-1} \right) - g_{\mu\nu} \text{Tr} \left(\Lambda_l B^{-1} \Lambda_{-l} B \right) \right) \quad (9.1)$$

and it is not traceless

$$\Theta_\mu^\mu = -2\kappa \text{Tr} \left(\Lambda_l B^{-1} \Lambda_{-l} B \right) \quad (9.2)$$

However, when the gradation (2.13) is performed by a grading operator Q_s , $\Theta_{\mu\nu}$ can be improved by adding the term

$$S_{\mu\nu} \equiv -\frac{\kappa}{2l} \text{Tr} \left(Q_s \left(\partial_\mu \left(B^{-1} \partial_\nu B \right) - g_{\mu\nu} \partial_\rho \left(B^{-1} \partial^\rho B \right) \right) \right) \quad (9.3)$$

Due to the fact B commutes with Q_s , such quantity is symmetric and it is trivially conserved

$$\partial^\mu S_{\mu\nu} = 0 \quad (9.4)$$

In addition, we have

$$S_\mu^\mu = \frac{\kappa}{2l} \text{Tr} \left(Q_s \partial_\mu \left(B^{-1} \partial^\mu B \right) \right) = 2\kappa \text{Tr} \left(\Lambda_l B^{-1} \Lambda_{-l} B \right) \quad (9.5)$$

where we have used the equations of motion (2.27), $x^\pm \equiv x \pm t$, and again the fact that B commutes with Q_s . Therefore, the improved energy-momentum tensor

$$T_{\mu\nu} \equiv \Theta_{\mu\nu} + S_{\mu\nu} \quad (9.6)$$

is conserved, symmetric and traceless.

The energy-momentum tensor of the G-AT models, defined by the equations (3.24), has a simple relation with the above tensor. Using the parameterization (3.16) of B one has

$$B^{-1} \partial B |_{\eta=0} = B_0^{-1} \partial B_0 + \partial \nu C \quad (9.7)$$

Moreover,

$$\begin{aligned} \Theta_{\mu\nu} = & \kappa \left(-\frac{1}{4} \text{Tr} \left(\partial_\mu B_0 \partial_\nu B_0^{-1} \right) + \frac{1}{8} g_{\mu\nu} \text{Tr} \left(\partial_\rho B_0 \partial^\rho B_0^{-1} \right) - g_{\mu\nu} \text{Tr} \left(e^{l\eta} \Lambda_l B_0^{-1} \Lambda_{-l} B_0 \right) \right) \\ & + \kappa \left(\frac{N_s}{4} (\partial_\mu \nu \partial_\nu \eta + \partial_\mu \eta \partial_\nu \nu - g_{\mu\nu} \partial_\rho \nu \partial^\rho \eta) \right), \end{aligned} \quad (9.8)$$

and

$$S_{\mu\nu} = -\frac{\kappa N_s}{2l} (\partial_\mu \partial_\nu \nu - g_{\mu\nu} \partial_\rho \partial^\rho \nu). \quad (9.9)$$

Then, one can easily verify that the canonical energy-momentum tensor for the G-AT models is $\Theta_{\mu\nu}^{\text{G-AT}} = \Theta_{\mu\nu} |_{\eta=0}$ and, so,

$$\begin{aligned} T_{\mu\nu} |_{\eta=0} &= \Theta_{\mu\nu}^{\text{G-AT}} + S_{\mu\nu} |_{\eta=0} \\ &= \Theta_{\mu\nu}^{\text{G-AT}} - \frac{\kappa}{2l} \text{Tr} \left(Q_s \left(\partial_\mu \left(\hat{B}_0^{-1} \partial_\nu \hat{B}_0 \right) - g_{\mu\nu} \partial_\rho \left(\hat{B}_0^{-1} \partial^\rho \hat{B}_0 \right) \right) \right) \\ &= \Theta_{\mu\nu}^{\text{G-AT}} - \frac{\kappa N_s}{2l} (\partial_\mu \partial_\nu \nu - g_{\mu\nu} \partial_\rho \partial^\rho \nu) \end{aligned} \quad (9.10)$$

where we have introduced the notation $\hat{B}_0 = B_0 e^{\nu C}$.

Now, following the reasoning of [9], suppose that we have a soliton like solution of the G-CAT model that can be put at rest in some Lorentz frame. Then the energy of such classical solution should be interpreted as the mass of the soliton. But since the theory is conformally invariant, it has no mass scale and therefore the soliton mass should vanish. If the solution is such that the η field vanishes then it is also a solution of the G-AT model; this theory is not conformally invariant and, therefore, the soliton mass evaluated with the G-AT energy-momentum tensor does not vanish. Using (9.10) one observes that the contribution to the soliton mass comes from a surface term. So, the soliton mass M is then

$$\begin{aligned} \frac{M v}{\sqrt{1-v^2}} &\equiv \int_{-\infty}^{\infty} dx \Theta_{10}^{\text{G-AT}} \\ &= \frac{\kappa}{2l} \text{Tr} \left(Q_s \hat{B}_0^{-1} \partial_t \hat{B}_0 \right) \Big|_{x=-\infty}^{x=\infty} = \frac{\kappa N_s}{2l} \partial_t \nu \Big|_{x=-\infty}^{x=\infty} \end{aligned} \quad (9.11)$$

where v is the soliton velocity ($c = 1$).

Next, using the explicit description of the integer gradations summarized in appendix B, we show that the trace in (9.11) can be expressed as an expectation value in a given representation. For an integral gradation (B.11), the generators of \hat{B}_0 are the Cartan subalgebra generators H_a^0 , $a = 1, 2, \dots, r$, the central term C and the step operators E_α^n whose grade is zero. So, we can write generically

$$\hat{B}_0^{-1} \partial_t \hat{B}_0 = \sum_{a=1}^r y_a H_a^0 + y_0 C + \text{terms in the direction of } E_\alpha^n \quad (9.12)$$

From (B.7)-(B.9)

$$\text{Tr} \left(Q_s \hat{B}_0^{-1} \partial_t \hat{B}_0 \right) = \sum_{a=1}^r \frac{2}{\alpha_a^2} y_a s_a + y_0 \sum_{i=0}^r s_i m_i^\psi. \quad (9.13)$$

Consider now the highest weight state $|\lambda_{s'}\rangle$ of an integrable representation of $\hat{\mathcal{G}}$ satisfying (B.17)-(B.21). Then we have

$$\langle \lambda_{s'} | \hat{B}_0^{-1} \partial_t \hat{B}_0 | \lambda_{s'} \rangle = \sum_{a=1}^r y_a s'_a + \frac{\psi^2}{2} y_0 \sum_{i=0}^r s'_i l_i^\psi \quad (9.14)$$

Therefore if we consider two different gradations \mathbf{s} and \mathbf{s}' such that⁵

$$s'_i = \frac{\psi^2}{\alpha_i^2} s_i \quad \text{for all } i = 0, \dots, r \quad (9.15)$$

then one gets

$$\frac{M v}{\sqrt{1-v^2}} = \frac{\kappa}{\psi^2 l} \langle \lambda_{s'} | \hat{B}_0^{-1} \partial_t \hat{B}_0 | \lambda_{s'} \rangle \Big|_{x=-\infty}^{x=\infty} \quad (9.16)$$

Notice that the subalgebras $\hat{\mathcal{G}}_0$ and $\hat{\mathcal{G}}'_0$ of zero grade with respect to the gradations \mathbf{s} and \mathbf{s}' , respectively, are equal. Therefore, $|\lambda_{s'}\rangle$ is an eigenstate of B_0 , and so

$$\frac{M v}{\sqrt{1-v^2}} = -\frac{\kappa}{\psi^2 l} \partial_t \ln \langle \lambda_{s'} | \hat{B}_0^{-1} | \lambda_{s'} \rangle \Big|_{x=-\infty}^{x=\infty} \quad (9.17)$$

From (5.15) and (5.14) one has, $\partial_t \ln \langle \lambda_{s'} | \hat{B}_0^{-1} | \lambda_{s'} \rangle \Big|_{x=-\infty}^{x=\infty} = \partial_t \ln \langle \lambda_{s'} | b^{-1} | \lambda_{s'} \rangle \Big|_{x=-\infty}^{x=\infty}$. As we have seen in (8.13), the soliton solutions correspond, in (7.26), to the choices $h_L = \tilde{h}_R = 1$ and $g = e^F$ with F being an eigenvector of $E_{\pm l}$. So, from (7.26), (8.12) and (8.15), it follows that

$$M = \kappa \frac{\gamma \sqrt{1-v^2}}{\psi^2 l} \frac{\langle \lambda_{s'} | e^{\gamma(x-vt)} F \exp(e^{\gamma(x-vt)} F) | \lambda_{s'} \rangle}{\langle \lambda_{s'} | \exp(e^{\gamma(x-vt)} F) | \lambda_{s'} \rangle} \Big|_{x=-\infty}^{x=\infty} \quad (9.18)$$

If there exists a positive integer $N'_{\max.}$ such that F satisfies

$$\langle \lambda_{s'} | F^{N'_{\max.}} | \lambda_{s'} \rangle \neq 0; \quad \text{for } N'_{\max.} > 0 \quad (9.19)$$

and also

$$\langle \lambda_{s'} | F^{n+1} | \lambda_{s'} \rangle = 0; \quad \text{for } n \geq N'_{\max.} \quad (9.20)$$

then, for $\gamma > 0$ only the upper limit in (9.18) contributes, and if $\gamma < 0$ only the lower one, so

$$M = \kappa \frac{N'_{\max.}}{\psi^2 l} |\gamma| \sqrt{1-v^2} = \kappa \frac{2}{\psi^2} \frac{N'_{\max.}}{l} \sqrt{-\omega_l \omega_{-l}} \quad (9.21)$$

where we used (8.15).

⁵Let us mention that these two gradations are equal, $\mathbf{s} = \mathbf{s}'$, if either the algebra \mathcal{G} is simply laced, or $s_i \neq 0$ only if α_i is a long root of \mathcal{G} .

Eq. (9.21) generalizes to the G-CAT models the mass formula obtained in [9, 20] for the solitons in the abelian Affine Toda models. However, a detailed analysis of each particular G-CAT model has to be done to ensure the physical significance of (9.21). The basic point of that analysis is to establish that the eigenvalues of $E_{\pm l}$, namely $\omega_{\pm l}$, are such that γ and v are real.

Notice that if we had taken in (7.6), $\eta = \eta_0 = \text{constant}$, then E_l would get multiplied by the factor v_η , introduced in (6.10). Therefore the eigenvalue ω_l , and consequently the soliton masses, would also get multiplied by the same factor. This is an evidence that the Higgs like mechanism discussed at the end of section 6 is also working in the generation of the soliton masses.

The mass formulas for the fundamental particles (6.9) and for the one-soliton solutions (9.21) indicate a very deep structure in such models, which is still to be understood. They might indicate a sort of duality similar to the electromagnetic duality conjectured by Montonen and Olive [25] in four dimensional gauge theories involving the interplay of monopoles and gauge particles. In fact, the use of the gradation \mathbf{s}' defined in (9.15), which is a technical artifact here to calculate the soliton masses, is an indication of the interplay between the algebra $\hat{\mathcal{G}}$ and its dual $\hat{\mathcal{G}}^v$ —roots interchanged by co-roots— similar to the electromagnetic duality. Some results concerning duality in Toda models were recently investigated in [26, 36].

Since F is an eigenvector of $E_{\pm l}$ then it is obviously an eigenvector of $[E_{-l}, [E_l, \cdot]]$. So, expanding it in eigenvectors of the grading operator $Q_{\mathbf{s}}$

$$F = \sum_n F^{(n)}; \quad [Q_{\mathbf{s}}, F^{(n)}] = nF^{(n)} \quad (9.22)$$

we observe that each component satisfies

$$[E_{-l}, [E_l, F^{(n)}]] = \omega_l \omega_{-l} F^{(n)} \quad (9.23)$$

Therefore if the zero mode $F^{(0)}$ is non-vanishing, we have a generator of $\hat{\mathcal{G}}_0$ with eigenvalue $\lambda = \omega_l \omega_{-l}$ and so, from (6.9) a fundamental particle of mass $m^2 = -4\omega_l \omega_{-l}$. So, for each F with non-vanishing zero mode we can put in correspondence a soliton and a fundamental particle with the masses satisfying

$$m_{\text{part.}} = \frac{l}{\kappa} \frac{\psi^2}{N'_{\text{max.}}} M_{\text{sol.}} \quad (9.24)$$

Obviously any two masses can be related by a proportionality constant. However, in the case of the abelian affine Toda models [20, 9], the integers $N'_{\text{max.}}$ for each species of particles and solitons, are such that the particle and soliton masses are proportional to the right and left Perron-Frobenius vectors respectively. This is what indicates the existence of a duality symmetry in the abelian affine Toda models. There remains to establish the types of non-abelian affine Toda models that present a similar relation. In any case, it is a remarkable fact that the soliton and particle masses are both determined by the eigenvalues of $E_{\pm l}$ in any one of these models.

10 Examples

Here, we work out in detail some aspects of the G-CAT models associated to the principal and to the homogeneous gradation of $\hat{\mathcal{G}}$, in which the dimension of $\hat{\mathcal{G}}_0$ is minimal and maximal, respectively. For the principal gradation, we recover the abelian CAT model of [6, 7], and we show that the Hirota's τ -functions defined in [9, 31] easily follow from the systematic approach of section 8. For the homogeneous gradation, we construct the one-soliton solutions when the finite Lie algebra \mathcal{G} is either compact or non-compact; in the former case, the resulting model has very suggestive features.

10.1 Principal gradation

The principal gradation of $\hat{\mathcal{G}}$ is defined by $s_i = 1$, $i = 0, 1, 2, \dots, r$ (see (B.12)). The highest weight state in the definition of the τ -function (8.1) is given by (B.24), *i.e.*,

$$|\lambda_{\text{ppal}}\rangle \equiv \bigotimes_{i=0}^r |\hat{\lambda}_i\rangle \quad (10.1)$$

From (B.18) we get

$$h_i |\lambda_{\text{ppal}}\rangle = |\lambda_{\text{ppal}}\rangle; \quad i = 0, 1, \dots, r \quad (10.2)$$

For the principal gradation the subalgebra $\hat{\mathcal{G}}_0$ is generated by h_i and D . So, we parameterize the group element b , in (8.1) as

$$b = \exp \left(\sum_{i=0}^r \varphi^i h_i \right) \quad (10.3)$$

In this case the zero mode $\tau_{\text{ppal}}^{(0)}$ (8.6) of the τ -function depends on all b fields. From (8.8) and (10.3)

$$\tau_{\text{ppal}}^{(0)} = b^{-1} |\lambda_{\text{ppal}}\rangle = e^{-\sum_{i=0}^r \varphi^i} |\lambda_{\text{ppal}}\rangle \quad (10.4)$$

We then define the Hirota's τ -functions, in this case, as

$$\tau_i \equiv \langle \lambda_{\text{ppal}} | e^{-\varphi^i h_i} | \lambda_{\text{ppal}} \rangle = \langle \hat{\lambda}_i | b^{-1} | \hat{\lambda}_i \rangle = e^{-\varphi^i} \quad (10.5)$$

and so

$$\varphi^i = -\ln \tau_i \quad i = 0, 1, 2, \dots, r \quad (10.6)$$

Therefore, the Hirota's τ -functions correspond to the components, in the tensor product space of the representation, of $\tau_{\text{ppal}}^{(0)}$. If we had parameterized b as

$$b = \exp \left(\sum_{a=1}^r \phi^a H_a^0 + \nu C \right) \quad (10.7)$$

we would get, since $\nu = \frac{2}{\psi^2} \varphi^0$ and $\phi^a = \varphi^a - l_a^\psi \varphi^0$, that

$$\nu = -\frac{2}{\psi^2} \ln \tau_0; \quad \phi^a = -\ln \frac{\tau_a}{(\tau_0)^{l_a^\psi}}; \quad a = 1, 2, \dots, r \quad (10.8)$$

This is exactly the definition of τ -function used in [9] to construct the soliton solutions of the abelian affine Toda models by the Hirota method (except for some terms associated to the fact that, in [9], the element $b_0 e^{\nu_0 C}$ was not factorized as in (5.15)).

For the case $l = 1$ one can take $\Lambda_{\pm l}$ in (2.22)-(2.23) to be

$$\Lambda_1 = \sum_{i=0}^r \bar{q}_i e_i; \quad \Lambda_{-1} = \sum_{i=0}^r q_i f_i \quad (10.9)$$

for some constants q_i and \bar{q}_i . The element b_0 in (5.11) can be taken as

$$b_0 = e^{\gamma \cdot H^0 + \rho D}; \quad \text{such that} \quad b_0 e_i b_0^{-1} \equiv \frac{l_i^\psi}{q_i \bar{q}_i} e_i \quad (10.10)$$

with l_a^ψ given in (B.22), and so

$$E_1 = b_0 \Lambda_1 b_0^{-1} = \sum_{i=0}^r \frac{l_i^\psi}{q_i} e_i; \quad E_{-1} = \Lambda_{-1} \quad (10.11)$$

satisfies

$$[E_1, E_{-1}] = \frac{2}{\psi^2} C \quad (10.12)$$

The symmetry groups given in (5.19), namely h_L and \tilde{h}_R , are trivial in this case, *i.e.*, they are exponentiations of the central term C . On the other hand, in the purely imaginary coupling constant regime, the transformations (5.20) are responsible for the infinitely degenerate vacua leading to the topological soliton solutions.

Therefore from (5.16) one gets the equations of motion for the abelian Conformal Affine Toda (CAT) models [6]

$$\partial_+ \partial_- \phi^a = -l_a^\psi \left(e^{K_{ab} \phi^b} - e^{K_{0b} \phi^b} \right) e^\eta \quad (10.13)$$

$$\partial_+ \partial_- \eta = 0 \quad (10.14)$$

$$\partial_+ \partial_- \nu = \frac{2}{\psi^2} \left(1 - e^{K_{0b} \phi^b} e^\eta \right) \quad (10.15)$$

and so $\phi^a = \eta = \nu = 0$ is a vacuum solution.

The soliton solutions and integrability properties of those models were extensively discussed in the literature [6, 9, 20, 8, 37] and we do not describe them here.

If one takes $l > 1$ one obtains different models. However, for the principal gradation, and for any gradation where all s_i 's are non-vanishing, the subalgebra $\hat{\mathcal{G}}_0$ is generated by h_i , $i = 0, 1, 2 \dots r$, and Q_s . Therefore, the generators of grade l are step operators which corresponding opposite root step operators have grade $-l$. Then, the difference of two roots of grade l is never a root. In this sense, the roots of grade l behave like a set of simple roots for some subalgebra of $\hat{\mathcal{G}}$, not necessarily simple. So, this indicates that by considering models with $l > 1$ for gradations where all s_i 's are non-vanishing one is going to get abelian Toda models (affine or not) for different subalgebras of $\hat{\mathcal{G}}$.

10.2 Homogeneous Gradation

In this case all s_i 's vanish except s_0 which equals one. So, the grading operator is just D

$$Q_{\text{hom.}} \equiv D \quad (10.16)$$

The subalgebra $\hat{\mathcal{G}}_0$ is generated by H_a^0 , $a = 1, 2, \dots, r$, $E_{\pm\alpha}^0$, D and C , *i.e.*, it is the simple algebra \mathcal{G} in addition to D and C . There is a great variety of choices for $E_{\pm l}$, but here we will consider those that can be rotated into the $H^{\pm l}$ subspace. So, we take

$$E_l = \sum_{a=1}^r q_a H_a^l \equiv q \cdot H^l; \quad E_{-l} = \sum_{a=1}^r \bar{q}_a H_a^{-l} \equiv \bar{q} \cdot H^{-l} \quad (10.17)$$

Therefore from (B.1) and (5.12) we have

$$\beta = l \sum_{a,b=1}^r q_a \eta_{ab} \bar{q}_b \equiv l q \cdot \bar{q} \quad (10.18)$$

The eigensubspaces $\hat{\mathcal{G}}_n$ of $Q_{\text{hom.}}$ are all copies of the adjoint representation of $\hat{\mathcal{G}}_0$, and therefore one does not expect the properties of the model to change by varying l . For instance, for any l , the symmetry groups of the model given in (5.19), are the same and equal to the Cartan subgroup of $\hat{\mathcal{G}}_0$, *i.e.*,

$$h_L \text{ and } \tilde{h}_R = \text{exponentiations of } H_a^0 \text{ for } a = 1, 2, \dots, r; \quad (10.19)$$

this in agreement with the comments below eq. (5.3). Notice that if q and/or \bar{q} are orthogonal to some root α , then the corresponding step operators $E_{\pm\alpha}^0$ are also generators of \tilde{h}_R and/or h_L . According to the discussion of section 6 the model has at least rank- $\mathcal{G}+1$ massless particles (depending upon the choice of q and \bar{q}). In fact, the diagonalization of the operator in (6.8) is quite simple (see (B.1)-(B.6))

$$[E_{-l}, [E_l, C]] = 0 \quad (10.20)$$

$$[E_{-l}, [E_l, H_a^0]] = 0 \quad (10.21)$$

$$[E_{-l}, [E_l, E_{\pm\alpha}^0]] = (q \cdot \alpha) (\bar{q} \cdot \alpha) E_{\pm\alpha}^0 \quad (10.22)$$

So, there is a mass degeneracy associated to each pair of roots $\pm\alpha$. In order to have non-negative masses one has, from (6.9), to take both q and $(-\bar{q})$ (or equivalently $(-q)$ and \bar{q}) lying in the Fundamental Weyl chamber [38]. However, such statement is misleading. Going back to the Klein-Gordon equation (6.5), one notices that the trace form multiplies the mass and the kinetic terms of the corresponding Lagrangian. Therefore if the trace form restricted to $\hat{\mathcal{G}}_0$ (more properly, to $\hat{\mathcal{G}}_0^*$) is not positive definite, we have a model where the energy is not bounded below. However, as we pointed out in sections 2 and 5, we can always choose the real form of the Kac-Moody algebra $\hat{\mathcal{G}}$ such that the trace form restricted to $\hat{\mathcal{G}}_0$ is positive definite. In the case of the homogeneous gradation we can take $\hat{\mathcal{G}}_0$ to be generated by H_a^0 , $E_\alpha^0 + E_{-\alpha}^0$, and $i(E_\alpha^0 - E_{-\alpha}^0)$, in addition to C and D . So, from (B.7)-(B.9) the trace form is positive definite except for the subspace generated by C and D . However,

since we are considering the model with the conformal symmetry spontaneously broken by $\eta = \eta_0 = \text{const.}$ (see (6.10)), one gets that the field in the direction of the central term does not contribute to the energy, since it is orthogonal to itself and to all remaining generators, see (5.3). Therefore we have a theory with masses and energies real and non-negative. We have $\text{rank-}\mathcal{G}+1$ massless particles and a number of massive particles equal to twice the number of positive roots and masses given by (see (6.9))

$$m_\alpha^2 = -4 (q \cdot \alpha) (\bar{q} \cdot \alpha); \quad \text{for any positive root } \alpha \text{ of } \mathcal{G} \quad (10.23)$$

with q and $(-\bar{q})$ being arbitrary vectors lying in the Fundamental Weyl chamber.

The highest weight state $|\lambda_s\rangle$ in this case is given by (see (B.24))

$$|\lambda_{\text{hom.}}\rangle \equiv |\hat{\lambda}_0\rangle \quad (10.24)$$

with $\hat{\lambda}_0$ given by (B.25). So, we have the scalar representation of $\hat{\mathcal{G}}_0$.

We parameterize the group element b introduced in (5.15) as

$$b = \tilde{b} e^{\frac{2}{\psi^2} \nu C} \quad (10.25)$$

Then, from (8.9) and (B.27) we get

$$\hat{\tau}_{\text{hom.}}^{(0)} = e^{-\nu}; \quad \text{or} \quad \tau_{\text{hom.}}^{(0)} = e^{-\nu} |\hat{\lambda}_0\rangle \quad (10.26)$$

Therefore, $\tau_{\text{hom.}}^{(0)}$ is not sufficient to parameterize the b -fields. In order to do that we need to consider more components of the τ -function.

We write the quantity $t^{(-l)}$ defined in (7.9) as

$$t^{(-l)} \equiv t_H^{(-l)} \cdot H^{-l} + \frac{1}{2} \sum_{\alpha > 0} \frac{\alpha^2}{2} \left(t_{\alpha, (1)}^{(-l)} (E_\alpha^{-l} + E_{-\alpha}^{-l}) + t_{\alpha, (2)}^{(-l)} (E_\alpha^{-l} - E_{-\alpha}^{-l}) / i \right) \quad (10.27)$$

Substituting that in (7.23), and using (10.17), (10.25) and (B.7)-(B.9) we get (fixing the gauge by $\tilde{h}_R = 1$)

$$\frac{2}{\psi^2} \partial_+ \nu = l q \cdot t_H^{(-l)} \quad (10.28)$$

$$\text{Tr} \left(\tilde{b}^{-1} \partial_+ \tilde{b} H_a^0 \right) = 0; \quad a = 1, 2, \dots, r \quad (10.29)$$

$$\text{Tr} \left(\tilde{b}^{-1} \partial_+ \tilde{b} (E_\alpha^0 + E_{-\alpha}^0) \right) = -i q \cdot \alpha t_{\alpha, (2)}^{(-l)} \quad (10.30)$$

$$\text{Tr} \left(\tilde{b}^{-1} \partial_+ \tilde{b} (E_\alpha^0 - E_{-\alpha}^0) / i \right) = i q \cdot \alpha t_{\alpha, (1)}^{(-l)} \quad (10.31)$$

We shall restrict ourselves to the solutions of the type described in (7.27). Therefore, from (8.6) and (8.10),

$$\tau_{\text{hom.}}^{(-s)} = 0; \quad 0 < s < l \quad (10.32)$$

and

$$\tau_{\text{hom.}}^{(-l)} = -e^{-\nu} t^{(-l)} |\hat{\lambda}_0\rangle \quad (10.33)$$

Following (8.18), we then define the Hirota τ -functions as

$$\tau_0 \equiv \langle \hat{\lambda}_0 | \tau_{\text{hom}} = e^{-\nu} \quad (10.34)$$

$$\tau_{\alpha,(1)} \equiv \langle \hat{\lambda}_0 | (E_\alpha^l + E_{-\alpha}^l) \tau_{\text{hom}} = -l \frac{\psi^2}{2} t_{\alpha,(1)}^{(-l)} \tau_0 \quad (10.35)$$

$$\tau_{\alpha,(2)} \equiv \langle \hat{\lambda}_0 | ((E_\alpha^l - E_{-\alpha}^l)/i) \tau_{\text{hom}} = -l \frac{\psi^2}{2} t_{\alpha,(2)}^{(-l)} \tau_0 \quad (10.36)$$

for every positive root α .

Notice there are no Hirota τ -functions associated to the b -fields in the direction of the Cartan subalgebra generators H_a^0 . The reason is that we have gauge fixed the symmetry of the model by setting in (10.19), $h_L = \tilde{h}_R = 1$. Indeed from (10.29), we see that $\tilde{b}^{-1} \partial_+ \tilde{b}$ has no component in the direction of the Cartan subalgebra.

The relation between the Hirota τ -functions and the fields of the model can be extracted from (10.34)-(10.36) and (10.28)-(10.31). In this case, such relations involve derivatives of the fields. The Hirota equations can then be obtained from the equations of motion for the fields. However we do not have to solve such Hirota equations, because the solutions have already been constructed by the dressing transformation method. In order to obtain the soliton solutions, we choose the constant group element g in (7.26) as in (8.13).

10.2.1 Non-compact solitons

As we have seen in section 8 the one-soliton solutions are associated to the eigenvectors of $E_{\pm l}$. In this case we have that

$$F_a^k(\zeta) = \sum_{n \in \mathbb{Z}} \zeta^{-nl} H_a^{k+nl}; \quad a = 1, 2, \dots, r; \quad k = 1, 2, \dots, l-1 \quad (10.37)$$

$$F_{\pm\alpha}^p(\zeta) = \sum_{n \in \mathbb{Z}} \zeta^{-nl} E_{\pm\alpha}^{p+nl}; \quad p = 0, 1, 2, \dots, l-1 \quad (10.38)$$

are eigenvectors of $E_{\pm l}$, given in (10.17)

$$[E_{\pm l}, F_a^k(\zeta)] = 0; \quad \text{for } k \neq 0 \quad (10.39)$$

$$[E_l, F_{\pm\alpha}^p(\zeta)] = \pm \zeta^l (q \cdot \alpha) F_{\pm\alpha}^p(\zeta) \quad (10.40)$$

$$[E_{-l}, F_{\pm\alpha}^p(\zeta)] = \pm \zeta^{-l} (\bar{q} \cdot \alpha) F_{\pm\alpha}^p(\zeta) \quad (10.41)$$

where ζ is a complex parameter.

Notice that if we take $k = 0$ in (10.37) we do not get zero eigenvectors, since

$$[E_l, F_a^0(\zeta)] = l \zeta^l \frac{2\alpha_a \cdot q}{\alpha_a^2} C; \quad [E_{-l}, F_a^0(\zeta)] = -l \zeta^{-l} \frac{2\alpha_a \cdot \bar{q}}{\alpha_a^2} C \quad (10.42)$$

One could choose q and \bar{q} to be orthogonal to all simple roots α_a except one, by taking them to lie in the direction of one of the fundamental weights. Therefore one could have rank- $\mathcal{G}-1$ eigenvectors of type F_a^0 with zero eigenvalue.

However the F 's with zero eigenvalue lead to constant solutions in (8.14), since from (8.15) we get that $\gamma = 0$. Therefore, there are no one-soliton solutions associated to F_a^k

given in (10.37). As we have commented on the paragraph below (7.27), we can not take F in (8.13) as one of the $F_{\pm\alpha}^p$ for $p \neq 0$, given in (10.38), because (10.32) will not be satisfied. So, we will construct for each $F_{\pm\alpha}^0$, a one-soliton solution.

Therefore from (8.14), (8.15), (10.40), (10.41) and (10.34)-(10.36) we get two one-soliton solutions for each positive root α , with the Hirota's τ -functions given by

$$\tau_0^{1-sol,\alpha} = 1 \quad (10.43)$$

$$\tau_{\alpha,(1)}^{1-sol,\alpha} = l \zeta^l \frac{\psi^2}{\alpha^2} e^{\gamma_\alpha(x-v_\alpha t)} \quad (10.44)$$

$$\tau_{\alpha,(2)}^{1-sol,\alpha} = i l \zeta^l \frac{\psi^2}{\alpha^2} e^{\gamma_\alpha(x-v_\alpha t)} \quad (10.45)$$

$$\tau_{\beta,(1)}^{1-sol,\alpha} = \tau_{\beta,(2)}^{1-sol,\alpha} = 0; \quad \text{for } \beta \neq \alpha \quad (10.46)$$

for g in (8.13), given by

$$g_{1-sol,\alpha}(\zeta) \equiv e^{F_\alpha^0(\zeta)} \quad (10.47)$$

and the parameters of the solitons being

$$\gamma_\alpha(\zeta) = (q\zeta^l - \bar{q}\zeta^{-l}) \cdot \alpha; \quad v_\alpha(\zeta) = -\frac{(q\zeta^l + \bar{q}\zeta^{-l}) \cdot \alpha}{(q\zeta^l - \bar{q}\zeta^{-l}) \cdot \alpha} \quad (10.48)$$

Notice that, for ζ real, we have $|v_\alpha| < 1$, since $(q \cdot \alpha)(\bar{q} \cdot \alpha) < 0$ for $\alpha > 0$ (q and $(-\bar{q})$ belong to the Fundamental Weyl chamber). The corresponding solution for the b -fields can be obtained from (10.28)-(10.31) and (10.34)-(10.36), giving ($\nu = 0$, see (10.25))

$$\tilde{b}_{1-sol,\alpha} = \exp\left(-e^{\gamma_\alpha(x-v_\alpha t)} E_\alpha^0\right) \quad (10.49)$$

Therefore, such one-soliton solution leads to an element of the non-compact real form of the group, as it “excites” only the field in the direction of E_α^0 .

In addition, for g in (8.13), given by

$$g_{1-sol,-\alpha}(\zeta) \equiv e^{F_{-\alpha}^0(\zeta)} \quad (10.50)$$

we have

$$\tau_0^{1-sol,-\alpha} = 1 \quad (10.51)$$

$$\tau_{\alpha,(1)}^{1-sol,-\alpha} = l \zeta^l \frac{\psi^2}{\alpha^2} e^{\gamma_{-\alpha}(x-v_{-\alpha} t)} \quad (10.52)$$

$$\tau_{\alpha,(2)}^{1-sol,-\alpha} = -i l \zeta^l \frac{\psi^2}{\alpha^2} e^{\gamma_{-\alpha}(x-v_{-\alpha} t)} \quad (10.53)$$

$$\tau_{\beta,(1)}^{1-sol,-\alpha} = \tau_{\beta,(2)}^{1-sol,-\alpha} = 0; \quad \text{for } \beta \neq \alpha \quad (10.54)$$

where

$$\gamma_{-\alpha}(\zeta) = -(q\zeta^l - \bar{q}\zeta^{-l}) \cdot \alpha; \quad v_{-\alpha}(\zeta) = -\frac{(q\zeta^l + \bar{q}\zeta^{-l}) \cdot \alpha}{(q\zeta^l - \bar{q}\zeta^{-l}) \cdot \alpha} \quad (10.55)$$

Again, we have $|v_{-\alpha}| < 1$ for ζ real and $(q \cdot \alpha)(\bar{q} \cdot \alpha) < 0$. Notice that for a given ζ , the one-soliton solutions associated to $F_\alpha^0(\zeta)$ and $F_{-\alpha}^0(\zeta)$ have the same velocity but opposite

width parameter γ (see (10.48) and (10.55)). The corresponding solution for the b -fields is similarly obtained from (10.28)-(10.31) and (10.34)-(10.36), giving ($\nu = 0$, see (10.25))

$$\tilde{b}_{1-sol, -\alpha} = \exp \left(-e^{\gamma - \alpha(x - v - \alpha t)} E_{-\alpha}^0 \right) \quad (10.56)$$

One can easily verify that (10.49) and (10.56) are indeed solutions, by substituting them directly into (5.16).

All these one-soliton solutions have vanishing masses. The reason is that, from (9.11), one observes that, only the component of $B_0^{-1} \partial_t B_0$ in the direction of the central term C contributes to the mass, since $Q \equiv D$ in this case. But, from (5.15), (10.25) (5.14), (10.43) and (10.51) that contribution is $\partial_t(\nu_0 + \frac{2}{\psi^2} \nu) C \sim t C$, and so, $\text{Tr} \left(Q B_0^{-1} \partial_t B_0 \right) \big|_{x=-\infty}^{x=\infty} = 0$. Another way of getting that is to use (9.18) and the fact that the expectation value of any non-vanishing positive power of $F_{\pm\alpha}^0(\zeta)$, in the highest weight state $|\hat{\lambda}_0\rangle$, is zero. Therefore, although we can put the one-soliton solutions in one to one correspondence with the massive fundamental particles, through the eigenvalues of $E_{\pm l}$, we do not have in this case, the proportionality of the masses as discussed in (9.24) ($N'_{\max.} = 0$). In the G-CAT model we do have massless solitons travelling with velocity $|v| < 1$. That is because the conformal symmetry does not allow a mass scale. It would be interesting to understand from a physical point of view why the above solitons are massless.

As we have said, the one-soliton solutions associated to the eigenvectors $F_{\pm\alpha}^{p+nl}$ for $p \neq 0$ can not be obtained by the above procedure due to the choice (10.32) we have made (see (7.27)). However, one does not expect them to correspond to new soliton solutions. The reason is that they do not exist in the case $l = 1$, and the equations of motion do not change considerably by changing l , as commented below eq.(5.3). As we pointed out above, the subspaces $\hat{\mathcal{G}}_n$ are copies of the adjoint representation of $\hat{\mathcal{G}}_0$ and the only equation of motion dependent on l (for $\eta = 0$) is that for the ν field. In addition, the eigenvectors $F_{\pm\alpha}^{p+nl}$ can be obtained from $F_{\pm\alpha}^{nl}$ by

$$F_{\pm\alpha}^{p+nl} = \pm \left[\frac{\alpha \cdot H^p}{\alpha^2}, F_{\pm\alpha}^{nl} \right]; \quad p = 1, 2, \dots, l-1 \quad (10.57)$$

and $\frac{\alpha \cdot H^p}{\alpha^2}$ lies in the centralizer of $E_{\pm l}$. This is in fact some sort of symmetry of the model. If T is an eigenvector of $E_{\pm l}$, $[E_{\pm l}, T] = \lambda_{\pm} T$, and if L lies in the centralizer of $E_{\pm l}$, then $[L, T]$ is also an eigenvector of $E_{\pm l}$ with the same eigenvalue.

10.2.2 Compact solitons

The one-soliton solutions for the compact real form of the group can be directly constructed in terms of the fields. Introducing the generators

$$T_1(\alpha) \equiv \frac{1}{2} (E_{\alpha} + E_{-\alpha}); \quad T_2(\alpha) \equiv \frac{1}{2i} (E_{\alpha} - E_{-\alpha}), \quad (10.58)$$

one can easily verify that the group elements

$$b^{(j)} \equiv e^{i\varphi T_j(\alpha)} e^{\nu C}; \quad j = 1, 2 \quad (10.59)$$

are solutions of the equations of motion (5.16), for $\eta = 0$ and $E_{\pm l}$ given by (10.17), if the fields φ and ν satisfy $(\partial_+ \partial_- = -\frac{1}{4} \partial^2 = -\frac{1}{4}(\partial_t^2 - \partial_x^2))$

$$\partial^2 \varphi = -m_\alpha^2 \sin \varphi \quad (10.60)$$

$$\partial^2 \nu = -\frac{l}{\alpha^2} m_\alpha^2 (\cos \varphi - 1) \quad (10.61)$$

where m_α^2 is given in (10.23).

Therefore, for every solution of the Sine-Gordon model we get two solutions, corresponding to $b^{(1)}$ and $b^{(2)}$ given in (10.59), of the G-AT model associated to the homogeneous gradation. The structure of these models is such that each $SU(2)$ subgroup associated to a positive root α can be considered independently.

We then have topological solitons, and the infinitely many vacua are associated to the invariance of the model under the transformations

$$b \rightarrow b e^{2\pi i n T_j(\alpha)}; \quad b \rightarrow e^{2\pi i n T_j(\alpha)} b \quad (10.62)$$

since the generators (10.17) satisfy

$$e^{2\pi i n T_j(\alpha)} E_{\pm l} e^{-2\pi i n T_j(\alpha)} = E_{\pm l} \quad (10.63)$$

Since we are working with the compact real form and b is parameterized as in (6.1), (10.64) implies a real transformation on the fields of the model.

The operators (10.17) are also certainly invariant under the transformations (5.20), which imply the fields transform as

$$b \rightarrow e^{2\pi i(\mu^v \cdot H^0 + n D)} b; \quad b \rightarrow b e^{2\pi i(\mu^v \cdot H^0 + n D)} \quad (10.64)$$

with μ^v a co-weight and n an integer. However, these are not relevant here, because the solutions we are considering do not possess fields in the direction of the Cartan subalgebra (except for ν); those have been gauge fixed by setting $h_L = \tilde{h}_R = 1$ in (10.19).

In order to evaluate the masses of the solutions (10.59) which can be put at rest in some Lorentz frame, we use the formula (9.11). Since the grading operator in this case is just D , and that is orthogonal to all generators except C , we get (see (5.15))

$$\frac{M v}{\sqrt{1-v^2}} = \frac{\kappa}{2l} \partial_t (\nu + \nu_0) \Big|_{x=-\infty}^{x=\infty} = \frac{\kappa}{2l} \partial_t \nu \Big|_{x=-\infty}^{x=\infty} \quad (10.65)$$

where we have used the fact that $\nu_0 \sim x^2 - t^2$ (see (5.14)).

The one-soliton solution of the Sine-Gordon model is

$$\varphi_{1-\text{sol}} = -4\epsilon_0 \arctan \exp \left(\frac{m_\alpha(x - vt - x_0)}{\sqrt{1-v^2}} \right) \quad (10.66)$$

where ϵ_0 is the topological charge and x_0 the initial position of the soliton. For the fundamental solitons, where $\epsilon_0 = \pm 1$, one gets from (10.61), (10.65) and (10.66) that the soliton masses are given by

$$M_\alpha^{\text{sol.}} = \kappa \frac{2}{\alpha^2} m_\alpha^{\text{part.}} \quad (10.67)$$

where $m_\alpha^{\text{part.}}$ is given in (10.23), and κ is the overall factor in the action (5.1).

Therefore for each positive root α we have two fundamental particles with the same mass $m_\alpha^{\text{part.}}$, and two one-soliton solutions, corresponding to $b^{(1)}$ and $b^{(2)}$ given by (10.59) and φ being the one-soliton of the S-G model, with equal masses $M_\alpha^{\text{sol.}}$. The relation between soliton and particle masses is given simply by (10.67). That is a remarkable relation. Like in the abelian affine Toda models, it indicates the existence of a duality transformation that interchanges the roles of particles and solitons. In addition, the factor $\frac{2}{\alpha^2}$ also indicates the corresponding interchange of the algebra and its dual.

The one-soliton solutions of this model have a very simple structure. For each generator $T_j(\alpha)$, ($j = 1, 2$), we have Sine-Gordon solitons excited independently. However, if one-solitons of different Sine-Gordon models are excited, they interact in a non-trivial way. The investigation of this last point, which is in progress, is of great interest because such model describes an integrable interaction of Sine-Gordon solitons.

11 Conclusions and discussion

We have constructed the Generalized Conformal Affine Toda models (G-CAT), and studied the most salient aspects of their structure. Along the text, we have remarked that most of these properties can be viewed as the generalization of similar features of the Abelian Conformal and of the Non-Abelian Affine Toda models. In our opinion, one of the most important points is that the Non-Abelian Affine Toda models can be understood as the result of the spontaneous breakdown of the conformal symmetry of the G-CAT models. Consequently, the masses of the solitons of the Affine Toda models are generated by a Higgs-like mechanism. Even more, this approach allows one to put in one-to-one correspondence the massive fundamental particles of the model with its one-soliton solutions; a relation which we expect to be relevant when quantizing these theories.

To study these G-CAT models, we have used different well known methods in the field of the integrable non-linear systems, and we have clarified the connection between all of them. Namely, in constructing the general solutions of the model, we have used and, when needed, we have suitably generalized the Leznov-Saveliev method for zero-curvature equations [2, 3, 15], the dressing-transformation approach [16, 17, 18, 19], and we have also defined the appropriated τ -functions [34, 35, 24]. The result is a systematic procedure for constructing the one-soliton solutions and an explicit formula for their masses.

Among the directions in which our work can be developed further, we point out the following. First, as we have seen in section 5, some of the left and right translations on the two-loop WZNW model remain as symmetries of the G-CAT model. However, as it happens in the abelian case, those which do not survive are twisted by the reduction procedure and should give rise to non-linear symmetries, *i.e.*, to W algebras. This has already been investigated in the abelian CAT models [39, 40, 8]. They have only two chiral remaining Kac-Moody currents in each chiral sector that, in fact, can be used to construct infinite W algebras; the other symmetries do not lead to chiral currents but to chiral charges [8]. Moreover, it would be interesting to investigate if some of the generators of the two-loop Virasoro algebra constructed in section A remain as symmetries of the G-CAT model.

Second, a systematic study of the different G-CAT models according to their specific properties has to be made. In particular, from the point of view of their eventual quantization, it would be particularly important to distinguish those models whose kinetic term is positive-definite, and whose action is real. Results in this direction have recently been presented in [29], and, concerning the first point, we have made some comments in sections 2 and 5. In section 5, we have also pointed out that a family of models admitting static solutions correspond to the choice of $E_{\pm l}$ as elements of some Heisenberg subalgebra of $\hat{\mathcal{G}}$. Then, the available classification of the Heisenberg subalgebras of an affine Kac-Moody algebra [30] can be used to construct a large number of particular models whose specific properties should be analysed.

Third, concerning the soliton solutions, their topological character should be investigated to establish in which cases a topological charge can be defined and also to understand its spectrum.

Finally, let us mention that the most interesting development to be made is the construction of the quantum theory of these models, which is expected to be integrable for a wide class of them. Then, it should be possible to construct an exact factorizable S-matrix and to compare the states of the resulting quantum theory with the solitons of the classical one.

Acknowledgements

We are grateful to M.V. Saveliev and A.V. Ramallo for helpful discussions. J.L.M. would like to thank Tim Hollowood for useful conversations. L.A.F. was partially supported by Ministerio de Educación y Ciencia (Spain) and FAPESP (Brazil). J.L.M. and J.S.G. have been supported partly by CICYT (AEN93-0729) and DGICYT (PB90-0772) (Spain).

A Appendix: The Two-Loop Virasoro algebra

The Sugawara construction for the two-loop Kac-Moody algebra (2.6)-(2.9) can be used to obtain an algebra much bigger than the Virasoro algebra. In order to do that we impose periodic boundary conditions on the currents. Namely, we take the space variable x being on a unit circle, and require the currents to satisfy, $J_a^m(x+2\pi) = J_a^m(x)$, $J^c(x+2\pi) = J^c(x)$ and $J^{\mathcal{D}}(x+2\pi) = J^{\mathcal{D}}(x)$. Since $J^c(x)$ commutes with $J_a^m(x)$ we can define a new set of currents by [13]

$$\tilde{J}_a^m(x) = J_a^m(x) \exp \left(\frac{m}{k} \sum_{n \neq 0} \frac{C_n}{n} e^{-inx} \right) \quad (\text{A.1})$$

where C_n are the modes of $J^c(x)$, *i.e.*,

$$J^c(x) = \sum_{n=-\infty}^{\infty} C_n e^{-inx} \quad (\text{A.2})$$

The zero mode C_0 is not included in the transformation (A.1) in order to respect the periodicity properties of $\tilde{J}_a^m(x)$. Notice that the transformation (A.1) is defined on a given

representation and not on the abstract algebra. Using (2.6)-(2.9) (with the redefinitions $f_{ab}^c \rightarrow i f_{ab}^c$ and $k \rightarrow ik$) one obtains [13]

$$[\tilde{J}_a^m(x), \tilde{J}_b^n(y)] = i f_{ab}^c \tilde{J}_c^{m+n}(x) \delta(x-y) + g_{ab} \delta_{m,-n} (i k \partial_x \delta(x-y) + m C_0 \delta(x-y)) \quad (\text{A.3})$$

$$[D_0, \tilde{J}_a^m(x)] = m \tilde{J}_a^m(x) \quad (\text{A.4})$$

$$[D_n, \tilde{J}_a^m(x)] = 0 \quad n \neq 0 \quad (\text{A.5})$$

where D_n are the modes of $J^{\mathcal{D}}(x)$, *i.e.*, $J^{\mathcal{D}}(x) = \sum_{n=-\infty}^{\infty} D_n e^{-inx}$. D_0 is the operator measuring the “momenta” m in an usual KM algebra [41]. Formulae (2.6)-(2.8) show that the J^c - $J^{\mathcal{D}}$ system is decoupled from the \tilde{J}' s. C_0 commutes with all the operators, *i.e.*, acts like a second central extension in the algebra of \tilde{J}' s.

Following the Sugawara construction we define

$$\mathcal{L}_m(x) \equiv \frac{1}{2} \sum_{a,b=1}^{\dim \mathcal{G}} \sum_{n=-\infty}^{\infty} g^{ab} \tilde{J}_a^{m+n}(x) \tilde{J}_b^{-n}(x) \quad (\text{A.6})$$

Using (A.3) one can check that

$$\begin{aligned} [\mathcal{L}_m(x), \mathcal{L}_n(y)] &= i k (2 \mathcal{L}_{m+n}(y) \delta'(x-y) - (\partial_y \mathcal{L}_{m+n}(y)) \delta(x-y)) \\ &+ C_0 (m-n) \mathcal{L}_{m+n}(y) \delta(x-y) \end{aligned} \quad (\text{A.7})$$

In addition, under $\mathcal{L}_m(x)$ the currents transform as

$$\begin{aligned} [\mathcal{L}_m(x), \tilde{J}_a^n(y)] &= i k \left(\tilde{J}_a^{m+n}(y) \delta'(x-y) - \partial_y \left(\tilde{J}_a^{m+n}(y) \right) \delta(x-y) \right) \\ &- n C_0 \tilde{J}_a^{m+n}(y) \delta(x-y) \end{aligned} \quad (\text{A.8})$$

Since the currents $\tilde{J}_a^m(x)$ satisfy periodic boundary conditions, so does $\mathcal{L}_m(x)$. We then expand it in modes as

$$\mathcal{L}_{m_1}(x) \equiv \sum_{m_2=-\infty}^{\infty} \mathcal{L}_{(m_1, m_2)} e^{-im_2 x} \quad (\text{A.9})$$

Using (A.7) one gets

$$[\mathcal{L}_{(m_1, m_2)}, \mathcal{L}_{(n_1, n_2)}] = (C_0 (m_1 - n_1) + k (m_2 - n_2)) \mathcal{L}_{(m_1+n_1, m_2+n_2)} \quad (\text{A.10})$$

where we have expanded the delta function as $\delta(x-y) = \sum_{n=-\infty}^{\infty} e^{-in(x-y)}$.

If the central terms C_0 and k are rational numbers, then one can always eliminate one of them from the above relation by a redefinition of the integer labels of $\mathcal{L}_{(m_1, m_2)}$. We introduce the $sl(2, \mathbb{Z})$ transformation

$$\begin{pmatrix} m'_1 \\ m'_2 \end{pmatrix} = M \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \quad (\text{A.11})$$

where $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$; with $a, b, c, d \in \mathbb{Z}$ and $\det M = 1$. The last condition is needed in order to make the transformation one to one. We then define the new generators

$$l_{(m_1, m_2)} \equiv \mathcal{L}_{(m'_1, m'_2)} \quad (\text{A.12})$$

From (A.10) and (A.11) we get

$$[l_{(m_1, m_2)}, l_{(n_1, n_2)}] = (C'_0 (m_1 - n_1) + k' (m_2 - n_2)) l_{(m_1 + n_1, m_2 + n_2)} \quad (\text{A.13})$$

where

$$(C'_0 k') = (C_0 k) M \quad (\text{A.14})$$

Now, if the central terms are rational, let us say $C_0 = p/q$ and $k = \bar{p}/\bar{q}$, then we can choose $a = q\bar{p}$ and $c = -p\bar{q}$, and b and c such that $dq\bar{p} + bp\bar{q} = 1$, which always has a solution. Therefore we get $C'_0 = 0$ and $k' = 1/q\bar{q}$.

Expanding the currents $\tilde{J}_a^m(x)$ in x , and making similar redefinitions in the pair of indices of its modes one can eliminate one of the central terms in (A.3) [13].

When the finite Lie algebra associated to the structure constants f_{ab}^c is compact, then it can be shown [13] that for unitary, highest weight representations for the currents $\tilde{J}_a^m(x)$ the central terms k and C_0 have to be integers [13]. In such a case the elimination procedure above works. The reduction procedure discussed in section 2 uses the Gauss decomposition. Therefore the two-loop Kac-Moody algebra (2.6)-(2.9) has to be based on a finite simple *non-compact* Lie algebra. In such cases k and C_0 are not necessarily integers and can in principle assume any value (even complex).

B Appendix: Affine Kac-Moody algebras

In the Chevalley basis, the commutation relations for an untwisted affine Kac-Moody algebra $\hat{\mathcal{G}}$ are

$$[H_a^m, H_b^n] = m C \eta_{ab} \delta_{m+n, 0} \quad (\text{B.1})$$

$$[H_a^m, E_\alpha^n] = \sum_{b=1}^r m_b^\alpha K_{ba} E_\alpha^{m+n} \quad (\text{B.2})$$

$$[E_\alpha^m, E_{-\alpha}^n] = \sum_{a=1}^r l_a^\alpha H_a^{m+n} + \frac{2}{\alpha^2} m C \delta_{m+n, 0} \quad (\text{B.3})$$

$$[E_\alpha^m, E_\beta^n] = (q+1) \epsilon(\alpha, \beta) E_{\alpha+\beta}^{m+n}; \quad \text{if } \alpha + \beta \text{ is a root} \quad (\text{B.4})$$

$$[D, E_\alpha^m] = m E_\alpha^m \quad (\text{B.5})$$

$$[D, H_a^m] = m H_a^m \quad (\text{B.6})$$

where $K_{ab} = 2\alpha_a \cdot \alpha_b / \alpha_b^2$ is the Cartan matrix of the finite simple Lie algebra \mathcal{G} associated to $\hat{\mathcal{G}}$ and generated by $\{H_a^0, E_\alpha^0\}$. $\eta_{ab} = \frac{2}{\alpha_a^2} K_{ab} = \eta_{ba}$, q is the highest positive integer such that $\beta - q\alpha$ is a root, $\epsilon(\alpha, \beta)$ are signs determined by the Jacobi identities, l_a^α and m_a^α are the integers in the expansions $\alpha/\alpha^2 = \sum_{a=1}^r l_a^\alpha \alpha_a / \alpha_a^2$ and $\alpha = \sum_{a=1}^r m_a^\alpha \alpha_a$, and $\alpha_1, \dots, \alpha_r$ are the simple roots of \mathcal{G} ($r \equiv \text{rank } \mathcal{G}$). $\hat{\mathcal{G}}$ has a symmetric non-degenerate bilinear form of $\hat{\mathcal{G}}$ which can be normalized as

$$\text{Tr}(H_a^m H_b^n) = \eta_{ab} \delta_{m+n, 0} \quad (\text{B.7})$$

$$\text{Tr}(E_\alpha^m E_\beta^n) = \frac{2}{\alpha^2} \delta_{\alpha+\beta, 0} \delta_{m+n, 0} \quad (\text{B.8})$$

$$\text{Tr}(C D) = 1 \quad (\text{B.9})$$

The integral gradations of $\hat{\mathcal{G}}$

$$\hat{\mathcal{G}} = \bigoplus_{n \in \mathbb{Z}} \hat{\mathcal{G}}_n \quad (\text{B.10})$$

have been classified in [27, 28]; the result is that every integer gradation with finite dimensional $\hat{\mathcal{G}}_0$ is conjugate to a gradation defined by a grading operator $Q_{\mathbf{s}}$ satisfying

$$[Q_{\mathbf{s}}, \hat{\mathcal{G}}_n] = n \hat{\mathcal{G}}_n ; \quad n \in \mathbb{Z}, \quad (\text{B.11})$$

and defined by

$$Q_{\mathbf{s}} = H_{\mathbf{s}} + N_{\mathbf{s}} D + \sigma C, \quad H_{\mathbf{s}} = \sum_{a=1}^r s_a \lambda_a^v \cdot H^0, \quad (\text{B.12})$$

where (s_0, s_1, \dots, s_r) is a vector of non-negative relatively prime integers, and $\lambda_a^v \equiv 2\lambda_a/\alpha_a^2$ with λ_a and α_a being the fundamental weights and simple roots of \mathcal{G} respectively. Moreover,

$$N_{\mathbf{s}} = \sum_{i=0}^r s_i m_i^{\psi}, \quad \psi = \sum_{a=1}^r m_a^{\psi} \alpha_a, \quad m_0^{\psi} \equiv 1 \quad (\text{B.13})$$

with ψ being the maximal root of \mathcal{G} ; obviously, the value of σ is arbitrary, but we shall make the standard choice $\sigma = -\text{Tr}(H_{\mathbf{s}}^2)/(2N_{\mathbf{s}})$, which ensures that $\text{Tr}(Q_{\mathbf{s}}^2) = 0$. Two gradations are equivalent if the corresponding vectors (s_0, s_1, \dots, s_r) and $(s'_0, s'_1, \dots, s'_r)$ are related by a symmetry of the extended Dynkin diagram of $\hat{\mathcal{G}}$. Therefore we have

$$[Q_{\mathbf{s}}, H_a^n] = n N_{\mathbf{s}} H_a^n \quad (\text{B.14})$$

$$[Q_{\mathbf{s}}, E_{\alpha}^n] = \left(\sum_{a=1}^r m_a s_a + n N_{\mathbf{s}} \right) E_{\alpha}^n \quad (\text{B.15})$$

The positive and negative simple root step operators of $\hat{\mathcal{G}}$ are $e_a \equiv E_{\alpha_a}^0$, $e_0 \equiv E_{-\psi}^1$ and $f_a \equiv E_{-\alpha_a}^0$, $f_0 \equiv E_{\psi}^{-1}$, and its Cartan subalgebra generators $h_a \equiv H_a^0$, $h_0 \equiv -\sum_{a=1}^r l_a^{\psi} H_a^0 + \frac{2}{\psi^2} C$, and D , with l_a^{ψ} given in (B.22)⁶; then, they satisfy

$$[Q_{\mathbf{s}}, h_i] = [Q_{\mathbf{s}}, D] = 0 ; \quad [Q_{\mathbf{s}}, e_i] = s_i e_i ; \quad [Q_{\mathbf{s}}, f_i] = -s_i f_i ; \quad i = 0, 1, \dots, r. \quad (\text{B.16})$$

A significant consequence of the existence of a grading operator is that $\text{Tr}(\hat{\mathcal{G}}_j \hat{\mathcal{G}}_k) = 0$ if $j + k \neq 0$, and that the subspaces $\hat{\mathcal{G}}_j$ and $\hat{\mathcal{G}}_{-j}$ are non-degenerately paired for all $j \in \mathbb{Z}$.

An important class of representations of the Kac-Moody algebras are the so called *integrable highest weight representations* [27]. They are defined in terms of a highest weight state $|\lambda_{\mathbf{s}}\rangle$ labelled by a gradation \mathbf{s} of $\hat{\mathcal{G}}$ (B.11). Such state is annihilated by the positive grade generators

$$\hat{\mathcal{G}}_+ |\lambda_{\mathbf{s}}\rangle = 0 \quad (\text{B.17})$$

⁶The positive integer numbers $a_i = m_i^{\psi}$ and $a_i^{\vee} = l_i^{\psi} \psi^2/2$, where $i = 0, 1, \dots, r$, are the Kac-labels and the dual Kac-labels of $\hat{\mathcal{G}}$, respectively [27], and $a_i = a_i^{\vee} \psi^2/\alpha_i^2$

and it is an eigenstate of all generators of the subalgebra $\hat{\mathcal{G}}_0$

$$h_i | \lambda_s \rangle = s_i | \lambda_s \rangle \quad (B.18)$$

$$f_i | \lambda_s \rangle = 0 ; \quad \text{for any } i \text{ such that } s_i = 0 \quad (B.19)$$

$$Q_s | \lambda_s \rangle = \eta_s | \lambda_s \rangle \quad (B.20)$$

$$C | \lambda_s \rangle = \frac{\psi^2}{2} \left(\sum_{i=0}^r l_i^\psi s_i \right) | \lambda_s \rangle \quad (B.21)$$

where l_i^ψ given by

$$\frac{\psi}{\psi^2} = \sum_{a=1}^r l_a^\psi \frac{\alpha_a}{\alpha_a^2} ; \quad l_0^\psi = 1. \quad (B.22)$$

It is always possible to modify the definition of the grading operator Q_s by adding a component in C in such a way that the eigenvalue η_s vanishes.

The integers m_i^ψ and l_i^ψ , given in (B.13) and (B.22), constitute respectively the left and right null vectors of the extended Cartan matrix of $\hat{\mathcal{G}}$

$$\sum_{i=0}^r m_i^\psi K_{ij} = 0 \quad \sum_{j=0}^r K_{ij} l_j^\psi = 0; \quad (B.23)$$

notice that $[h_i, e_j] = K_{ji} e_j$ for all $i, j = 0, 1, \dots, r$.

The highest weight states $| \lambda_s \rangle$ can be realized as

$$| \lambda_s \rangle \equiv \bigotimes_{i=0}^r | \hat{\lambda}_i \rangle^{\otimes s_i} \quad (B.24)$$

where $| \hat{\lambda}_i \rangle$ are the highest weight states of the fundamental representations of $\hat{\mathcal{G}}$, and $\hat{\lambda}_i$ are the corresponding fundamental weights of $\hat{\mathcal{G}}$. Namely [41]

$$\hat{\lambda}_0 = (0, \psi^2/2, 0) \quad (B.25)$$

$$\hat{\lambda}_a = (\lambda_a, l_a^\psi \psi^2/2, 0) \quad (B.26)$$

where λ_a , $a = 1, 2, \dots, r$ are the fundamental weights of the finite Lie algebra \mathcal{G} associated to $\hat{\mathcal{G}}$, l_a^ψ is defined in (B.22), and the entries are the eigenvalues of H_a^0 , C and D respectively, *i.e.*,

$$H_a^0 | \hat{\lambda}_0 \rangle = 0; \quad C | \hat{\lambda}_0 \rangle = \frac{\psi^2}{2} | \hat{\lambda}_0 \rangle \quad (B.27)$$

$$H_b^0 | \hat{\lambda}_a \rangle = \delta_{a,b} | \hat{\lambda}_a \rangle; \quad C | \hat{\lambda}_a \rangle = \frac{\psi^2}{2} l_a^\psi | \hat{\lambda}_a \rangle \quad (B.28)$$

and

$$D | \hat{\lambda}_i \rangle = 0 \quad (B.29)$$

Notice that in each of the $r + 1$ fundamental representations of $\hat{\mathcal{G}}$, the (unique) highest weight state satisfies

$$h_j | \hat{\lambda}_i \rangle = \delta_{ij} | \hat{\lambda}_i \rangle \quad (\text{B.30})$$

$$e_j | \hat{\lambda}_i \rangle = 0; \quad \text{for any } j \quad (\text{B.31})$$

$$f_j | \hat{\lambda}_i \rangle = 0; \quad \text{for } j \neq i \quad (\text{B.32})$$

$$f_i^2 | \hat{\lambda}_i \rangle = 0. \quad (\text{B.33})$$

Therefore the generators e_i and f_i are nilpotent when acting on (B.24), and these representations are actually integrable.

C Appendix: A particular basis of $\hat{\mathcal{G}}_0$

Let us consider the integer gradation associated to a vector (s_0, s_1, \dots, s_r) defined by the grading operator

$$Q_{\mathbf{s}} = H_{\mathbf{s}} + N_{\mathbf{s}} D - \frac{1}{2N_{\mathbf{s}}} \text{Tr}(H_{\mathbf{s}}^2) C, \quad (\text{C.1})$$

where

$$H_{\mathbf{s}} = \sum_{a=1}^r s_a \lambda_a^v \cdot H^0, \quad (\text{C.2})$$

according to eq.(B.12). $H_{\mathbf{s}}$ is an element of the Cartan subalgebra of the finite Lie algebra \mathcal{G} (that one with zero grade w.r.t. D), and it defines a finite integer gradation of \mathcal{G}

$$\mathcal{G} = \bigoplus_{j=-L}^{+L} \mathcal{G}_j, \quad [H_{\mathbf{s}}, v] = j v \quad \text{for all } v \in \mathcal{G}_j. \quad (\text{C.3})$$

Notice that the step operator corresponding to the maximal root has to be an element of maximal grade, *i.e.*, $E_{\psi} \in \mathcal{G}_L$; but the grade of E_{ψ} can be obtained using eq.(B.15), which, taking into account (B.13) too, implies that

$$L = \sum_{a=1}^r s_a m_a^{\psi} = N_{\mathbf{s}} - s_0. \quad (\text{C.4})$$

Therefore, in addition to C and D , the zero graded subspace of $\hat{\mathcal{G}}$ will contain \mathcal{G}_0^0 when $s_0 \neq 0$, or $\mathcal{G}_0^0 \oplus \mathcal{G}_{N_{\mathbf{s}}}^{-1} \oplus \mathcal{G}_{-N_{\mathbf{s}}}^{+1}$ when $s_0 = 0$; notice that we have introduced the notation $\mathcal{G}_k^m = \{u^m \in \hat{\mathcal{G}} \mid u \in \mathcal{G}_k\}$.

In the case where $s_0 = 0$, the commutator of two elements of $\hat{\mathcal{G}}_0$ may produce terms in the direction of the central term C . Therefore, in order to establish the relation between the Conformal Affine (G-CAT) and the Affine (G-AT) Toda models, it is convenient to choose a basis of the subspace $\hat{\mathcal{G}}_0$ in which the affine components \mathcal{G}_0^0 and $\mathcal{G}_{\pm N_{\mathbf{s}}}^{\mp 1}$ are orthogonal to the other components of $\hat{\mathcal{G}}_0$; this only requires to change the basis of the subalgebra of $\hat{\mathcal{G}}_0$

generated by C , D , and the Cartan subalgebra of \mathcal{G}_0 . Taking into account the eqs.(B.7)-(B.9), a convenient choice is $\{C, Q_s, \widetilde{H}_1^0, \dots, \widetilde{H}_r^0\}$, where

$$\begin{aligned}\widetilde{H}_a^0 &= H_a^0 - \frac{1}{N_s} \text{Tr}(H_s H_a^0) C \\ &= H_a^0 - \frac{2}{\alpha_a^2} \frac{s_a}{N_s} C,\end{aligned}\tag{C.5}$$

which satisfies

$$\text{Tr}(C^2) = \text{Tr}(C \widetilde{H}_a^0) = \text{Tr}(Q_s^2) = \text{Tr}(Q_s \widetilde{H}_a^0) = 0,\tag{C.6}$$

$$\text{Tr}(Q_s C) = N_s, \quad \text{Tr}(\widetilde{H}_a^0 \widetilde{H}_b^0) = \text{Tr}(H_a^0 H_b^0) = \eta_{ab},\tag{C.7}$$

for all $a, b = 1 \dots, r$.

Now, let $\hat{\mathcal{G}}_0^*$ be the subspace of $\hat{\mathcal{G}}_0$ generated by $\{\widetilde{H}_1^0, \dots, \widetilde{H}_r^0\}$ and by either $\{E_\alpha^0 \mid E_\alpha \in \mathcal{G}_0\}$, if $s_0 \neq 0$, or $\{E_\alpha^0, E_{\pm\beta}^{\mp 1} \mid E_\alpha \in \mathcal{G}_0, E_{\pm\beta} \in \mathcal{G}_{\pm N_s}\}$, if $s_0 = 0$. Then, $\hat{\mathcal{G}}_0^*$ is a subalgebra of $\hat{\mathcal{G}}_0$, and it is isomorphic either to \mathcal{G}_0^0 , when $s_0 \neq 0$, or to $\mathcal{G}_0^0 \oplus \mathcal{G}_{-N_s}^1 \oplus \mathcal{G}_{+N_s}^{-1}$, if $s_0 = 0$. Moreover, by construction, $\hat{\mathcal{G}}_0^*$ commutes with Q_s and C ; therefore, any element $u \in \hat{\mathcal{G}}_0$ can be parameterized as

$$u = u^* + \nu C + \eta Q_s, \quad u^* \in \hat{\mathcal{G}}_0^*,\tag{C.8}$$

where

$$\nu = \frac{1}{N_s} \text{Tr}(u Q_s), \quad \eta = \frac{1}{N_s} \text{Tr}(u C).\tag{C.9}$$

Correspondingly, any element of the group formed by exponentiating the elements of $\hat{\mathcal{G}}_0$ can be expressed as

$$B = B_0 \exp(\nu C + \eta Q_s),\tag{C.10}$$

where B_0 is an element of the subgroup obtained by exponentiating the subalgebra $\hat{\mathcal{G}}_0^*$. It is worth noticing that trace form Tr , restricted to $\hat{\mathcal{G}}_0^*$, is also non-degenerate. Finally, let us point out that, when $s_0 \neq 0$, B_0 is the field of the non-abelian affine Toda model associated to the affine Lie algebra $\hat{\mathcal{G}}$ and to the gradation induced by Q_s . However, when $s_0 = 0$, we have models of non-abelian affine type which, as far as we know, were not considered in the literature before.

In order to substitute eq.(C.10) in eq. (5.1), the following expressions will be useful:

$$\text{Tr}(\partial_\mu B \partial^\mu B^{-1}) = \text{Tr}(\partial_\mu B_0 \partial^\mu B_0^{-1}) - 2 N_s \partial_\mu \nu \partial^\mu \eta,\tag{C.11}$$

$$\epsilon^{ijk} \text{Tr}(B^{-1} \partial_i B B^{-1} \partial_j B B^{-1} \partial_k B) = \epsilon^{ijk} \text{Tr}(B_0^{-1} \partial_i B_0 B_0^{-1} \partial_j B_0 B_0^{-1} \partial_k B_0),\tag{C.12}$$

$$\text{Tr}(\Lambda_l B^{-1} \Lambda_{-l} B) = e^{l \eta} \text{Tr}(\Lambda_l B_0^{-1} \Lambda_{-l} B_0).\tag{C.13}$$

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